Quaternions and Applications

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1 Introduction and Historical Overview

In 1805 a truly remarkable mathematician was born in Dublin, Ireland. This man eventually became one of the most influential mathematicians of the 19th century. He was Sir William Hamilton. This childhood prodigy would have a profound influence on many fields of mathematics and physical sciences. In this paper his greatest legacy is going to be explored: that of quaternions. Hamilton believed that his invention of the quaternion; a hypercomplex set of numbers representing space-time; held the key to the future of mathematical physics. Although Hamilton's belief in his invention never wavered, it was not until the middle of this century that his achievement was recognized as having true merit, with the advent of quantum mechanics and special relativity. Although quaternions hasn't and probably never will become the predominant mathematical method of physics; his early work and the invention of quaternions influenced and fostered the modern vectorial calculus we use today. In these and many other fields Hamilton's works still influence today and most certainly will into the next century.

In order to appreciate the nature of Hamilton's invention we must understand the state of mathematics, physics, and science as it existed before and during Hamilton's lifetime. In the late 18^{th} and into the early 19^{th} centuries' mathematicians became interested in representing "directed lines of force" in physical problems in some other fashion other than Euclidean geometry. By this time the concept of a complex number was established and understood by most mathematicians although it was treated more as a mathematical oddity than a worthwhile pursuit of study. Caspar Wessel was the first to recognize the use of complex numbers in representing two dimensional space. Many others also developed this area. These works prompted many to try to extend their use to three dimensional space. Well before Hamilton ever became interested in this problem no fewer than ten other mathematicians had tried to find a triplet set and failed to propose a satisfactory system. Hamilton also attempted this problem and failed. However through this early work Hamilton for the first time rationalized complex numbers as ordered pairs of real numbers rather than geometrically. This was very satisfying to the scientific community as many mathematicians did not like geometric proofs, but preferred algebraic ones. Hamilton thus became interested in the study of complex numbers and what we would call today as dimensional analysis.

Around this time Hamilton also devised a general set of equations of motion for multi parameter systems of motion (Hamiltonian theory: "On a General Method in Dynamics"). Through this work and his failure to solve the triplet problem Hamilton began to explore what he called time development. Basically Hamilton, being a bit of a metaphysicist, believed that time evolution was an integral, nay indistinguishable part of any system involving motion. This prompted Hamilton to include time as a component of a directed line of force system of motion and in effect he began to look for a four dimensional system instead of a triplet algebra. On October 16, 1843 Hamilton solved this problem and stated the equation that forms the basis of his theory: $i^2 = j^2 = k^2 = ijk = -1$. So momental was this flash of brilliance the he felt compelled to carve this equation on the Brougham Bridge. Hamilton was a prodigious writer and wrote everywhere on anything suitable, in this case while walking with his wife on the bridge and a pocket knife was handy.

Hamilton quickly developed a paper on his discovery and presented it to the Royal Irish Society during the first general meeting of the session that November. Hamilton however took ten years to publish a book on the subject but he submitted many papers to journals advocating the importance of his discovery. "Lecture on Quaternions" appeared in 1853 and became his first true mathematical book. This 500+ page book was well received but sales were moderate. However mathematicians did not become zealot believers in his quaternion theory due to the fact that "Lecture" is long, difficult, and included many complicated notations and terms representing various quaternion types and operations. The only other mathematician that truly shared Hamilton's vision of the quaternion future of mathematics was Peter Tait. Tait was a Scottish classmate of Maxwell at Edinburgh University. He ordered Hamilton's book in 1858 only as the title caught his eye in a publisher's book list. Tait proceeded to go through the first six chapters and asked Thomas Andrews to write Hamilton asking if Tait could correspond with him. Tait's first letter to Hamilton was a flattering introduction to Hamilton, however in the letter Tait claimed that he got through most of the book easily and quickly. Hamilton did not really believe this claim but he returned the letter. This became the beginning of a correspondence that lasted for the rest of Hamilton's life. Through this interaction Hamilton came to recognize Tait's mastery of the subject and Tait became a student and contemporary in the subject of quaternions. In fact Tait became the most influential promoter of quaternions of the late 19th century.

Although Hamilton still worked on other areas of study and ran the Dunsink Observatory, he believed that quaternions were important enough that he devoted 22 years of his life until his death in 1865 on the subject. The majority of this labour was devoted to a second book meant to be a practical guide on quaternions use. Tait was also writing a book on the subject but since he used unpublished results given to him by Hamilton he asked permission to publish. Hamilton agreed as long as Tait waited for him to publish his unfinished book. Tait followed his mentor's request and after Hamilton's son published "Elements of Quaternions" in 1866 posthumously, Tait published his "Treatise on Quaternions". Although Hamilton intended this second book to be a guide it was longer than "Lectures" and was unfinished. Tait's book however is clear, concise, and complete and thus became the actual guide on quaternions to which most interested scientists both then and now refer.

During the last half of Hamilton's life, and to the turn of the century, the groundwork that leads us to modern vector calculus was being laid. Grassmann, Gibbs and Heaviside were developing results, and notations (some of it borrowed from Hamilton) and applying it to problems in electromagnetism and motion. Maxwell, being a friend of Tait, was introduced to quaternions and in his famous 1873 paper on electromagnetism; he used his famous results and included quaternion forms as well. However due to Hamilton slowness of publishing, and the complexity of his works, quaternions were not recognized as being very practical. The Gibbs-Heaviside forms are easier to use and appear to be a shortcut to a workable mathematics. It was not until the appearance of relativity and quantum mechanics that the weakness of the Gibbs-Heaviside system was recognized. About this time quaternions, as other of Hamilton's mathematical theories, were truly being looked at as being potentially useful. Some of his results were introduced into the modern vector system but quaternion forms were not used. The strangeness of the quaternion forms can not displace the more familiar vector forms that we rely on today.

Although Hamilton's influence is still being felt in modern theory today, his belief in the superiority of his quaternions has not been realized. It is important to note that Hamilton did however create the first algebra that covered three dimensions and in doing so was the first to abandon an algebraic law (commutativity). This encouraged others to leave the stringent rules and customs of the past mathematics which led to our modern algebra and calculus. It is possible to extract all modern vector calculus from the quaternion theory. In fact modern vectors may be viewed as what Hamilton called the vector part of a quaternion. Hamilton was also the first to suggest the notation, use, and operations of the Del operator ∇, recognizing its importance in the field of physics. Hamilton also gave us many names and showed the use of many of the components and operations of vector calculus such as scalar, vector, vector (cross) product, and the scalar (dot) product. He also showed their important use in solving physical problems. Please also note that in this paper we will be reverse engineering much of the quaternion theory from the modern calculus. Thus throughout the work it will be assumed that the reader has some familiarity with vector calculus. It is to be hoped that the reader will gain some appreciation of the practical use of the quaternion.

2 Real Quaternions and Rotations

2-1 Definition and Algebra of Quaternions

Through out this work it will be assumed that the reader has some familiarity with vector calculus. A normal vector (3-vector) may be denoted by $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ where $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors in Cartesian coordinates. Let's explore the algebra of quaternions, define the quaternion as $q_1 = a + \vec{A}$, where: a is a real number and $\vec{A}$ is a 3-vector.

Let us also define $q_2 = b + \vec{B}$ as above with $b \neq a$ and $\vec{A} \neq \vec{B}$.

The sum and difference is as one would expect; the "scalar" parts are operated on separately as are the "vector" parts:

$$\text{Sum: } q_1 + q_2 = \left(a + \vec{A}\right) + \left(b + \vec{B}\right) = \left(a + b\right) + \left(\vec{A} + \vec{B}\right) \qquad \text{2-1-1}$$

$$\text{Difference: } q_1 - q_2 = \left(a - b\right) + \left(\vec{A} - \vec{B}\right) \qquad \text{2-1-2}$$

The product needs to be looked at. Try finding 4 terms under normal multiplication:

$$q_1 q_2 = \left(a + \vec{A}\right)\left(b + \vec{B}\right) = ab + a\vec{B} + b\vec{A} + \vec{A}\vec{B} \qquad \text{2-1-3}$$

clearly, $a\vec{B} = a\left(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}\right)$ and similarly for $b\vec{A}$.

A form for $\vec{A}\vec{B}$ must then be found that makes sense mathematically:

$$\begin{aligned}\vec{A}\vec{B} &= \left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)\left(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}\right) \\ &= \hat{i}^2 A_1B_1 + \hat{j}^2 A_2B_2 + \hat{k}^2 A_3B_3 + \left(\hat{j}\hat{k}A_2B_2 + \hat{k}\hat{j}A_3B_2\right) \\ &\quad + \left(\hat{k}\hat{i}A_3B_1 + ikA_1B_3\right) + \left(\hat{i}\hat{j}A_1B_2 + \hat{i}\hat{j}A_2B_1\right)\end{aligned} \qquad \text{2-1-4}$$

Following Hamilton; define: $\hat{i}^2 = -1 \quad \hat{i}\hat{j} = \hat{k} = -\hat{j}\hat{i}$

$$\hat{j}^2 = -1 \quad \hat{j}\hat{k} = \hat{i} = -\hat{j}\hat{k} \qquad \text{2-1-5}$$

$$\hat{k}^2 = -1 \quad \hat{k}\hat{i} = \hat{j} = -\hat{k}\hat{i}$$

So

$$\begin{aligned}\vec{A}\vec{B} = &-\left(A_1B_1 + A_2B_2 + A_3B_3\right) + \hat{i}\left(A_2B_3 - A_3B_2\right) \\ &+ \hat{j}\left(A_3B_1 - A_1B_3\right) + \hat{k}\left(A_1B_2 - A_2B_1\right)\end{aligned} \qquad \text{2-1-6}$$

Clearly the product of $\vec{A}$ and $\vec{B}$ must be $\vec{A}\vec{B} = -\left(\vec{A} \circ \vec{B}\right) + \vec{A} \times \vec{B}$. This is known as the Hamiltonian Product and it is very important in quaternion algebra.

So a quaternion is defined as $q_1 = a + \vec{A} = a + A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and makes sense mathematically under the Hamiltonian product. Three identities of the Hamiltonian product are shown below:

$$\vec{A}\vec{A} = -\left(\vec{A} \circ \vec{A}\right) + \vec{A} \times \vec{A} = -\left(A_1^2 + A_2^2 + A_3^2\right) = -\left|\vec{A}\right|^2 \qquad \text{2-1-7}$$

If $\hat{n}$ is a unit vector, $\hat{n}^2 = -1$ 2-1-8

$$\begin{aligned}\vec{A}\vec{B} + \vec{B}\vec{A} &= -\vec{A} \circ \vec{B} + \vec{A} \times \vec{B} - \vec{B} \circ \vec{A} + \vec{B} \times \vec{A} \\ &= -2\vec{A} \circ \vec{B} + \vec{A} \times \vec{B} - \vec{A} \times \vec{B} \\ &= -2\vec{A} \circ \vec{B} \qquad \text{2-1-9a} \\ \therefore\ \vec{A} \circ \vec{B} &= -\frac{1}{2}\left(\vec{A}\vec{B} + \vec{B}\vec{A}\right)\end{aligned}$$

$$\vec{A}\vec{B} - \vec{B}\vec{A} = -\vec{A} \circ \vec{B} + \vec{A} \times \vec{B} + \vec{B} \circ \vec{A} - \vec{B} \times \vec{A}$$
$$= -\vec{A} \times \vec{B} \qquad \text{2-1-9b}$$
$$\therefore \vec{A} \times \vec{B} = -\frac{1}{2}\left(\vec{A}\vec{B} - \vec{B}\vec{A}\right)$$

One other useful vector identity is:

$$\begin{aligned} \vec{A} &= -\hat{n}^2\vec{A} \\ &= -\hat{n}\hat{n}\vec{A} = -\hat{n}\left(\hat{n}\vec{A}\right) \\ &= -\hat{n}\left(-\hat{n} \circ \vec{A} + \hat{n} \times \vec{A}\right) \\ &= \left(\vec{A} \circ \hat{n}\right)\hat{n} + \hat{n}\left(\vec{A} \times \hat{n}\right) \qquad \text{2-1-10} \\ &= \left(\vec{A} \circ \hat{n}\right)\hat{n} + \left(-\left(\vec{A} \times \hat{n}\right) \circ \hat{n} + \hat{n} \times \left(\vec{A} \times \hat{n}\right)\right) \\ &= \left(\vec{A} \circ \hat{n}\right)\hat{n} + \hat{n} \times \left(\vec{A} \times \hat{n}\right) \\ &= \left(\vec{A} \circ \hat{n}\right)\hat{n} + \hat{n} \times \vec{A} \times \hat{n} \end{aligned}$$

Note: 1) since n is a unit vector: $\hat{n} \times \left(\vec{A} \times \hat{n}\right) = \left(\vec{A} \times \hat{n}\right) \times \hat{n} = \hat{n} \times \vec{A} \times \hat{n}$

(brackets are redundant).

2) $\vec{A} \times \hat{n} = 0$

This gives the parallel and perpendicular components of the vector.

2-2 The Conjugate, Modulus, and Inverse of Quaternions

The Conjugate: As quaternions are a set of hypercomplex numbers, we need to look at conjugation and absolute values before considering division. If $q = a + \vec{A}$ the conjugate is defined as $q^{\times} = a - \vec{A}$

Consider the product:

$$\begin{aligned} qq^{\times} &= (a+\vec{A})(a-\vec{A}) = a^2 - \vec{A}\vec{A} \\ &= a^2 - \left(A_1^{\,2}\hat{i}^2 + A_2^{\,2}\hat{j}^2 + A_3^{\,2}\hat{k}^2\right) \\ &= a^2 + A_1^{\,2} + A_2^{\,2} + A_3^{\,2} \\ &= q^{\times}q \end{aligned} \qquad \text{2-2-1}$$

which is real and positive definite (positive and zero if and only if q is zero).

Now we can define the modulus of q:

$$|q| = \sqrt{qq^{\times}} = \sqrt{q^{\times}q} = \sqrt{a^2 + A_1^{\,2} + A_2^{\,2} + A_3^{\,2}} \qquad \text{2-2-2}$$

The effect of conjugation on a product: consider

$$\begin{aligned} (\vec{A}\vec{B})^{\times} &= (-\vec{A}\circ\vec{B} + \vec{A}\times\vec{B})^{\times} \\ &= -(\vec{A}^{\times})\circ(\vec{B}^{\times}) - (\vec{A}^{\times})\times(\vec{B}^{\times}) \\ &= -(-\vec{A})\circ(-\vec{B}) - (-\vec{A})\times -(\vec{B}) \\ &= -\vec{A}\circ\vec{B} + \vec{A}\times(-\vec{B}) \\ &= -\vec{B}\circ\vec{A} - (-\vec{B})\times\vec{A} \\ &= -\vec{B}\circ\vec{A} + \vec{B}\times\vec{A} \\ &= -\vec{A}\circ\vec{B} - \vec{A}\times\vec{B} \\ &= \vec{B}^{\times}\vec{A}^{\times} \end{aligned} \qquad \text{2-2-3}$$

Since the scalar part is not affected and the vector part changes sign the order is reversed under conjugation.

The multiplicative inverse: Let $q^{-1} = \frac{1}{|q|^2}q^{\times}$ then $q^{-1}q = \left(\left(\frac{1}{|q|^2}\right)q^{\times}\right)q = \frac{|q|^2}{|q|^2} = 1 = qq^{-1}$ as expected for multiplicative inverse.

Thus the previous two sections have shown that for a real quaternion defined as $q = a + \vec{A}$ all algebraic operations are defined. Thus quaternions form an algebra.

2-3 Euler's Formula and Quaternions

Euler's formula is $e^{i\theta} = \cos\theta + i\sin\theta$ which can be used to represent complex numbers. Thus there must be an Euler form for the quaternions.

Using $q = a + \vec{A}$ which we can also write as $q = a + A\hat{n}$ where $A = |\vec{A}| = \sqrt{A_1^{\,2} + A_2^{\,2} + A_3^{\,2}}$ and $\hat{n}$ is a unit vector in the direction A is pointing (or $\hat{n} = \frac{1}{A}\vec{A}$), we can look for the polar form of q: If $r = |q| = \sqrt{a^2 + A^2}$

Then we can define $\cos\theta = \dfrac{a}{r}$ and $\sin\theta = \dfrac{A}{r}$

so $a = r\cos\theta$ and $A = r\sin\theta$

Then $q = r\cos\theta + r\sin\theta\hat{r}$

$$= r(\cos\theta + \hat{r}\sin\theta) \quad (\text{since } \hat{r}^2 = -1)$$
$$= re^{\hat{r}\theta}$$

where $e^{\hat{r}\theta}$ is a unimodular quaternion.

2-4 Unimodular Quaternions as a Group

There are four properties which must be satisfied for the unimodular quaternions to be a group: closure, associativity, existence of an identity, and existence of inverses.

First: let set G of elements a, b, .. ∋ a, b, c, … ∈ G be unimodular quaternions.

$$i.e. \quad a = e^{\hat{a}\theta} = 1 + \vec{A} = 1 + \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3 \qquad 2\text{-}4\text{-}1$$

Since the unimodular quaternions are a subset of the quaternions, two of these properties are true; namely associativity and the existence of an identity (namely 1). These have been seen for quaternions under multiplication but not explicitly proved. In fact they are included in our definition of the quaternion.

Existence of Inverses: here the elements of the set a, b, c all have the property $|a| = 1$ since they are unimodular and thus $a^{-1} = a^{\times} = e^{-\hat{r}\theta}$ which is an inverse.

Then

$$\begin{aligned} aa^{\times} &= e^{\hat{r}\theta}e^{-\hat{r}\theta} \\ &= e^{\hat{r}\theta-\hat{r}\theta} \\ &= e^{0} \\ &= 1 \end{aligned} \qquad 2\text{-}4\text{-}2$$

as required for the inverse.

Closure: Given a, b we need their product to be unimodular (in the set G).

$$\begin{aligned} ab(ab)^{\times} &= ab(b^{\times}a^{\times}) \\ &= a1a^{\times} \quad \text{still unimodular} \\ &= 1 \end{aligned} \qquad 2\text{-}4\text{-}3$$

or if

$$\begin{aligned} a &\equiv e^{\hat{r}_1\theta_1} \\ b &\equiv e^{\hat{r}_2\theta_2} \end{aligned} \qquad 2\text{-}4\text{-}4$$

Then $ab = e^{\hat{r}_1\theta_1} e^{\hat{r}_2\theta_2} = e^{\hat{r}_3\theta_3}$ also unimodular 2-4-5

Thus all four properties are satisfied and the elements of set G are indeed a group. The following properties are also shared:

-identity of the group is unique

-inverse of each element of the group is unique

-a b = a c then b = c → dual (i.e. b a = c a then b = c).

2-5 The Triangle Inequality

Before showing the quaternion form of the Triangle Inequality we need to prove $\forall q_1, q_2,\ |q_1 q_2| = |q_1||q_2|$. Let $q_1 = r_1 e^{\hat{r}_1\theta_1}$ and $q_2 = r_2 e^{\hat{r}_2\theta_2}$ then

$$
\begin{aligned}
|q_1 q_2| &= |q_1||q_2| \\
|r_1 e^{\hat{r}_1\theta_1} r_2 e^{\hat{r}_2\theta_2}| &= |r_1 e^{\hat{r}_1\theta_1}||r_2 e^{\hat{r}_2\theta_2}| \\
|r_1 r_2| &= |r_1||r_2| \\
\mathrm{Re}(q_1 q_2) &= \mathrm{Re}(q_1)\mathrm{Re}(q_2) \\
r_1 r_2 &= r_1 r_2 \\
LS &= RS
\end{aligned}
\qquad \text{2-5-1}
$$

Now the Triangle Inequality: $|q_1 + q_2| \le |q_1| + |q_2|$.

$$
\begin{aligned}
|q_1 + q_2|^2 &= (q_1 + q_2)(q_1^{\times} + q_2^{\times}) \\
&= q_1 q_1^{\times} + q_2 q_2^{\times} + q_1 q_2^{\times} + q_2 q_1^{\times} \\
&= |q_1|^2 + |q_2|^2 + q_1 q_2^{\times} + (q_1 q_2^{\times})^{\times} \\
&= |q_1|^2 + |q_2|^2 + 2\,\mathrm{Re}(q_1 q_2^{\times}) \\
&\le |q_1|^2 + |q_2|^2 + 2|q_1 q_2| \\
&\le |q_1|^2 + |q_2|^2 + 2|q_1||q_2| \\
&\le (|q_1| + |q_2|)^2
\end{aligned}
\qquad \text{2-5-2}
$$

$$so\ |q_1 + q_2| \le |q_1| + |q_2|$$

2-6 Rotations in a Plane Using Complex Numbers

Given the x-y plane as in figure 2-6-1. Let a plane vector be represented by $\vec{A} = A_1\hat{i} + A_2\hat{j}$ and the angle between the vector and the x-axis be θ.

The vector $\vec{A}'$ represents the new position of $\vec{A}$ after rotation through ϕ degrees about the origin (active transformation).

NOTE: 1) positive rotation is clockwise, i.e. this system is right-handed.

2) this is an active transformation due to the fact that the axis remains fixed.

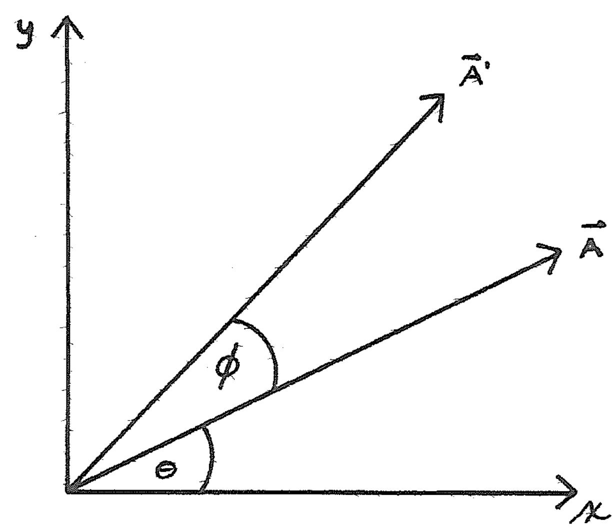


Figure 2-6- 1

In order to find the components of $\vec{A}'$ namely A_1' and A_2' it is easier to use polar co-ordinates.

Thus vector $\vec{A}$ may be represented by:

$$\begin{aligned} A_1 &= r\cos\theta \\ A_2 &= r\sin\theta \\ r &= |\vec{A}| = \sqrt{A_1^{\,2} + A_2^{\,2}} \end{aligned} \qquad 2\text{-}6\text{-}1$$

The rotation simply increases the angle between the x-axis and the vector $\vec{A}'$ to be $\theta + \phi$ and leaves the length of the vector, r unchanged. Thus the components may be found as

$$A_1' = r\cos(\theta+\phi) \qquad \text{and} \qquad A_2' = r\sin(\theta+\phi)$$

$$\begin{aligned} &= r(\cos\theta\cos\phi - \sin\theta\sin\phi) && = r(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= (r\cos\theta)\cos\phi - (r\sin\theta)\sin\phi && = A_2\cos\phi + A_1\sin\phi \end{aligned} \qquad 2\text{-}6\text{-}2$$

$$= A_1\cos\phi - A_2\sin\phi$$

or in matrix form: $$\begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \qquad 2\text{-}6\text{-}3$$

Now if we note the form of vectors represented in section 2-3, we recognize that vector $\vec{A}$ may also be written as: $\underline{A} = A_1 + iA_2 = r(\cos\theta + i\sin\theta) = re^{i\theta}$ which is a complex form.

Now the previous rotation may be more elegantly written as:

$$\underline{A}' = re^{i(\theta+\phi)} = e^{i\phi}re^{i\theta} = e^{i\phi}\underline{A} \qquad 2\text{-}6\text{-}4$$

To see that this represents the work shown previously note that equation 2-6-4 may also be written as

$$\begin{aligned} A_1' + iA_2' &= e^{i\phi\theta}(A_1 + iA_2) \\ &= (\cos\phi + i\sin\phi)(A_1 + iA_2) \\ &= \cos\phi A_1 + i\cos\phi A_2 + i\sin\phi A_1 - \sin\phi A_2 \end{aligned} \qquad 2\text{-}6\text{-}5$$

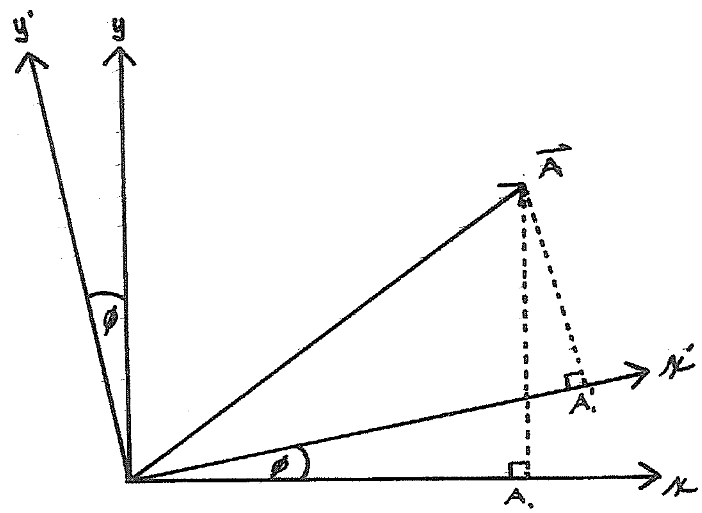
and if the real and imaginary parts are separated:

Real: $A_1' = \cos\phi A_1 - \sin\phi A_2$ Imaginary: $A_2' = \cos\phi A_2 + \sin\phi A_1$

as before but obtained in one equation much more easily.

Now a <u>passive</u> rotation in which the vector remains fixed, and the axis moves, is illustrated in figure 2-6-2. This rotation can also be viewed as an active rotation through -ϕ degrees. Thus the equation representing this is

$$\underline{A}' = re^{i(\theta-\phi)} = e^{-i\phi}\underline{A} \qquad 2\text{-}6\text{-}6$$



Example 2-6-1: An active rotation

Using complex numbers, rotate vector $\vec{A} = \hat{i} + \sqrt{3}\hat{j}$ through 60 degrees in the x-y plane.

Solution: $60^\circ = \frac{\pi}{3}$ counterclockwise $\Rightarrow$ $e^{i\frac{\pi}{3}}$

$$
\begin{aligned}
\vec{A}' &= \vec{A}e^{i\frac{\pi}{2}} = \vec{A}\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right) \\
&= \left(1+\sqrt{3}i\right)\left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) \\
&= \frac{3+3\sqrt{3}i - 4i - 4\sqrt{3}(-1)}{2} \\
&= \frac{3-4\sqrt{3}}{2} + \frac{i\left(3\sqrt{3}-4\right)}{2} \\
&= \frac{3-4\sqrt{3}}{2}\hat{i} + \frac{\left(3\sqrt{3}-4\right)}{2}\hat{j}
\end{aligned}
\qquad \text{2-6-7}
$$

2-7 Active Rotations in 3-Space using Real Quaternions

In order to extend the use of complex numbers to 3-space, quaternions may be used since they already have a unimodular form, and represent 3-space. It is also advantageous to find a formula for the rotation of a vector about an arbitrary axis of $\hat{n}$ direction in 3-diamensions.

Try $\vec{A}' = e^{\hat{n}\theta}\vec{A}$ 2-7-1

which will work only if $\vec{A}$ is perpendicular to $\hat{n}$.

To see why consider the case where $\hat{n}$ is parallel to $\vec{A}$ (i.e. $\vec{A} = A\hat{n}$) then $\vec{A}$ should remain unchanged:

$$
\begin{aligned}
\vec{A}' &= e\vec{A} = \left(\cos\theta + \hat{n}\sin\theta\right)A\hat{n} \\
&= A\cos\theta\hat{n} + A\sin\theta\hat{n}^2 \\
&= -A\sin\theta + A\cos\theta\hat{n}
\end{aligned}
\qquad \text{2-7-2}
$$

which is no longer a vector!

We need to look for some properties of $\vec{A}$ that $\vec{A}'$ must also share. The length of $\vec{A}'$ must be the same as $\vec{A}$. So $\left|\vec{A}\right|^2 = \vec{A}^{\times}\vec{A} = \left|\vec{A}'\right|^2 = \vec{A}'^{\times}\vec{A}'$ 2-7-3 but note $\vec{A}^{\times} = -\vec{A}$ 2-7-4

referring back to $\vec{A}' = e^{\hat{n}\theta}\vec{A}$

$$\vec{A}'^{\times} = \vec{A}^{\times}\left(e^{\hat{n}\theta}\right)^{\times} = -\vec{A}e^{-\hat{n}\theta} \qquad 2\text{-}7\text{-}5$$

and thus $\vec{A}'^{\times} = -\vec{A}'$

$$-\vec{A}e^{-\hat{n}\theta} = -e^{\hat{n}\theta}\vec{A} \qquad 2\text{-}7\text{-}6$$

which is not equal in general.

However the fact that the length must remain constant leads us to believe that the above formula, equation 2-7-3, is a guide to the correct form. Look at the form $\vec{A}' = R\vec{A}R^{\times}$ where R is a factor like $e^{\hat{n}\theta}$. Apply this through the equivalent length requirement of equation 2-7-3:

$$\begin{aligned}(\vec{A}')^{\times} &= \left(R\vec{A}R^{\times}\right)^{\times} \\ &= R^{\times\times}\vec{A}^{\times}R^{\times} \\ &= -R\vec{A}R^{\times} \\ &= -\vec{A}'\end{aligned} \qquad 2\text{-}7\text{-}7$$

Note the reversal of order and the vector is unchanged as required!

But what is R? Equation 2-7-1 must represent the effect of R and $R^{\times}$ on the vector rotated through θ degrees. This leads us to a form of R from equation 2-7-3. Let us move half of the factor $e^{\hat{n}\theta}$ through the vector respecting conjugation:

$$
\begin{aligned}
\vec{A}' &= e^{\hat{n}\theta}\vec{A} \\
&= e^{\frac{\hat{n}\theta}{2}}e^{\frac{\hat{n}\theta}{2}}\left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right) \\
&= e^{\frac{\hat{n}\theta}{2}}\left(A_1 e^{\frac{\hat{n}\theta}{2}}\hat{i} + A_2 e^{\frac{\hat{n}\theta}{2}}\hat{j} + A_3 e^{\frac{\hat{n}\theta}{2}}\hat{k}\right) \\
&= e^{\frac{\hat{n}\theta}{2}}\left(A_1\left(\cos\tfrac{\theta}{2} + \hat{n}\sin\tfrac{\theta}{2}\right)\hat{i} + A_2\left(\cos\tfrac{\theta}{2} + \hat{n}\sin\tfrac{\theta}{2}\right)\hat{j} + A_3\left(\cos\tfrac{\theta}{2} + \hat{n}\sin\tfrac{\theta}{2}\right)\hat{k}\right) \\
&= e^{\frac{\hat{n}\theta}{2}}\left(A_1\left(\cos\tfrac{\theta}{2}\hat{i} + \hat{n}\hat{i}\sin\tfrac{\theta}{2}\right) + A_2\left(\cos\tfrac{\theta}{2}\hat{j} + \hat{n}\hat{j}\sin\tfrac{\theta}{2}\right) + A_3\left(\cos\tfrac{\theta}{2}\hat{k} + \hat{n}\hat{k}\sin\tfrac{\theta}{2}\right)\right) \qquad \text{2-7-8} \\
&= e^{\frac{\hat{n}\theta}{2}}\left(A_1\left(\hat{i}\cos\tfrac{\theta}{2} - \hat{i}\hat{n}\sin\tfrac{\theta}{2}\right) + A_2\left(\hat{j}\cos\tfrac{\theta}{2} - \hat{j}\hat{n}\sin\tfrac{\theta}{2}\right) + A_3\left(\hat{k}\cos\tfrac{\theta}{2} - \hat{k}\hat{n}\sin\tfrac{\theta}{2}\right)\right) \\
&= e^{\frac{\hat{n}\theta}{2}}\left(A_1\hat{i}e^{-\frac{\hat{n}\theta}{2}} + A_2\hat{j}e^{-\frac{\hat{n}\theta}{2}} + A_3\hat{k}e^{-\frac{\hat{n}\theta}{2}}\right) \\
&= e^{\frac{\hat{n}\theta}{2}}\left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)e^{-\frac{\hat{n}\theta}{2}} \\
&= e^{\frac{\hat{n}\theta}{2}}\vec{A}e^{-\frac{\hat{n}\theta}{2}}
\end{aligned}
$$

Thus an active rotation in the arbitrary direction $\hat{n}$ is given by

$$\vec{A}' = e^{\frac{\hat{n}\theta}{2}}\vec{A}e^{-\frac{\hat{n}\theta}{2}} \qquad \text{2-7-9.}$$

Let us check equation 2-7-9 under equation 2-7-3 and the case where $\vec{A}$ and $\hat{n}$ are parallel.

First:

$$
\begin{aligned}
\left|\vec{A}'\right|^2 &= \vec{A}'^{\times}\vec{A} = -\vec{A}'\vec{A}' \\
&= -e^{\hat{n}\frac{\theta}{2}}\vec{A}e^{-\hat{n}\frac{\theta}{2}}e^{\hat{n}\frac{\theta}{2}}\vec{A}e^{-\hat{n}\frac{\theta}{2}} \\
&= -e^{\hat{n}\frac{\theta}{2}}\vec{A}(1)\vec{A}e^{-\hat{n}\frac{\theta}{2}} \\
&= e^{\hat{n}\frac{\theta}{2}}\left(-\vec{A}\vec{A}\right)e^{-\hat{n}\frac{\theta}{2}} \qquad \text{2-7-10} \\
&= e^{\hat{n}\frac{\theta}{2}}\left|\vec{A}\right|e^{-\hat{n}\frac{\theta}{2}} \\
&= \left|\vec{A}\right|^2 e^{\hat{n}\frac{\theta}{2}}e^{-\hat{n}\frac{\theta}{2}} \\
&= \left|\vec{A}\right|^2
\end{aligned}
$$

which is fine.

And

$$\begin{aligned}
\vec{A}' &= e^{\hat{n}\frac{\theta}{2}}\vec{A}e^{-\hat{n}\frac{\theta}{2}} \\
&= e^{\hat{n}\frac{\theta}{2}}A\hat{n}e^{-\hat{n}\frac{\theta}{2}} \\
&= A\hat{n}e^{\hat{n}\frac{\theta}{2}}e^{-\hat{n}\frac{\theta}{2}} \\
&= A\hat{n} \\
&= \vec{A}
\end{aligned} \qquad \text{2-7-11}$$

$\vec{A}$ is left unchanged as required. Thus equation 2-7-9 is indeed correct.

A quick shortcut: We can make use of vector behavior by separating $\vec{A}$ into perpendicular and parallel components:

parallel is the projection of $\vec{A}$ on $\hat{n}$: $\vec{A}_{II} = \left(\vec{A} \circ \hat{n}\right)\hat{n}$ 2-7-12a

perpendicular is what is left: $\vec{A}_{\perp} = \vec{A} - \vec{A}_{II}$ 2-7-12b

Such that $\vec{A} = \vec{A}_{II} + \vec{A}_{\perp}$. 2-7-13

The parallel component is left unchanged by the rotation and the perpendicular component can be handled by this simple formula: $\vec{A}_{\perp}' = e^{\hat{n}\theta}\vec{A}_{\perp}$ 2-7-14

Thus the general rotation may also be found from: $\vec{A}' = \vec{A}_{II} + e^{\hat{n}\theta}\vec{A}_{\perp}$ 2-7-15

2-8 Illustrative Examples

a) Rotate the vector $\vec{A} = 2\hat{i} + 2\hat{j} + \hat{k}$ through 90° about an axis pointing in the direction $\vec{B} = 4\hat{j} + 3\hat{k}$.

<u>Solution</u>: we need a directional unit vector $\hat{n} = \frac{\vec{B}}{|\vec{B}|} = \frac{4\hat{j} + 3\hat{k}}{\sqrt{0+16+9}} = \frac{4}{5}\hat{j} + \frac{3}{5}\hat{k}$

now apply equation 2-7-9:

$$\vec{A}' = e^{\frac{\hat{n}\theta}{2}} \vec{A} e^{-\frac{\hat{n}\theta}{2}} = \left(\cos\tfrac{\pi}{4} + \hat{n}\sin\tfrac{\pi}{4}\right)\vec{A}\left(\cos\tfrac{\pi}{4} - \hat{n}\sin\tfrac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2}(1+\hat{n})\vec{A}\frac{\sqrt{2}}{2}(1-\hat{n}) = \frac{1}{2}\left[(\vec{A} + \hat{n}\vec{A})(1-\hat{n})\right]$$

$$= \frac{1}{2}\left[\vec{A} - \vec{A}\hat{n} + \hat{n}\vec{A} - \hat{n}\vec{A}\hat{n}\right]$$

$$= \frac{1}{2}\left[\vec{A} - \vec{A}\frac{1}{5}(4\hat{j}+3\hat{k}) + \frac{1}{5}(4\hat{j}+3\hat{k})\vec{A} - \frac{1}{5}(4\hat{j}+3\hat{k})\vec{A}\frac{1}{5}(4\hat{j}+3\hat{k})\right]$$

$$= \frac{1}{50}\left[25\vec{A} - 5\vec{A}(4\hat{j}+3\hat{k}) + 5(4\hat{j}+3\hat{k})\vec{A} - (4\hat{j}+3\hat{k})\vec{A}(4\hat{j}+3\hat{k})\right]$$

$$= \frac{1}{50}\begin{bmatrix} 25\vec{A} - 5(2\hat{i}+2\hat{j}+\hat{k})(4\hat{j}+3\hat{k}) + 5(4\hat{j}+3\hat{k})(2\hat{i}+2\hat{j}+\hat{k}) \\ -(4\hat{j}+3\hat{k})(2\hat{i}+2\hat{j}+\hat{k})(4\hat{j}+3\hat{k}) \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} 25\vec{A} - 5(11 + 2\hat{i} - 6\hat{j} + 8\hat{k}) - 5(-11 + 2\hat{i} - 6\hat{j} + 8\hat{k}) \\ -(4\hat{j}+3\hat{k})(11 + 2\hat{i} - 6\hat{j} + 8\hat{k}) \end{bmatrix}$$

$$= \frac{1}{50}\left[25\vec{A} - 10(2\hat{i} - 6\hat{j} + 8\hat{k}) - (44\hat{j} + 33\hat{k} + 50\hat{i} + 6\hat{j} - 8\hat{k})\right]$$

$$= \frac{1}{50}\left[25(2\hat{i}+2\hat{j}+\hat{k}) - 10(2\hat{i} - 6\hat{j} + 8\hat{k}) - (50\hat{i} + 50\hat{j} - 25\hat{k})\right]$$

$$= \frac{1}{50}\left[2(25-10-25)\hat{i} + 2(25+30-25)\hat{j} + (25-80+25)\hat{k}\right]$$

$$= \frac{1}{25}\left[-10\hat{i} + 30\hat{j} - 15\hat{k}\right]$$

b) Using the Shortcut: Rotate $\vec{A} = 12\hat{i} - 3\hat{j} - 9\hat{k}$ through 37° $\left(\theta = \cos^{-1} 0.8\right)$ about $\vec{B} = 2\hat{i} + 2\hat{j} + \hat{k}$.

Solution: $\hat{n} = \frac{\vec{B}}{|\vec{B}|} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{4+4+1}} = \frac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$

$$\vec{A}_{II} = \left[\left(12\hat{i} - 3\hat{j} - 9\hat{k}\right) \circ \frac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)\right]\frac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$$
$$= [8 - 2 - 3]\frac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$$
$$= \left(2\hat{i} + 2\hat{j} + \hat{k}\right)$$

$$\vec{A}_{\perp} = \left(12\hat{i} - 3\hat{j} - 9\hat{k}\right) - \left(2\hat{i} + 2\hat{j} + \hat{k}\right) = \left(10\hat{i} - 5\hat{j} - 10\hat{k}\right) A_{II} + e^{\hat{n}\theta}\vec{A}_{\perp}$$
$$= \vec{A}_{II} + \cos\left(\cos^{-1} 0.8\right) + \hat{n}\sin\left(\cos^{-1} 0.8\right)\vec{A}_{\perp}$$
$$= \left(2\hat{i} + 2\hat{j} + \hat{k}\right) + \left(\frac{8}{10} + \frac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)\frac{6}{10}\right)\left(10\hat{i} - 5\hat{j} - 10\hat{k}\right)$$
$$= \left(2\hat{i} + 2\hat{j} + \hat{k}\right) + \left(\frac{8}{10} + \frac{1}{5}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)\right)5\left(2\hat{i} - \hat{j} - 2\hat{k}\right)$$
$$= \left(2\hat{i} + 2\hat{j} + \hat{k}\right) + \left(\left(8\hat{i} - 4\hat{j} - 8\hat{k}\right) + \left((4 - 2 - 2) + (-4 + 1)\hat{i} + (2 + 4)\hat{j} + (-2 - 4)\hat{k}\right)\right)$$
$$= \left(2\hat{i} + 2\hat{j} + \hat{k}\right) + \left(\left(8\hat{i} - 4\hat{j} - 8\hat{k}\right) + \left(0 - 3\hat{i} + 6\hat{j} - 6\hat{k}\right)\right)$$
$$= \left(2\hat{i} + 2\hat{j} + \hat{k}\right) + \left(5\hat{i} + 2\hat{j} - 14\hat{k}\right)$$
$$= 7\hat{i} + 4\hat{j} - 13\hat{k}$$

3 Complex Quaternions and Special Relativity

3-1 Definition of a Complex Quaternion

A complex quaternion extends the algebra of quaternions to include imaginary quantities as part of the quaternion. In order to designate the real and imaginary parts of the complex quaternion we must introduce a fourth square root of -1; as i; which commutes with $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ i.e. $i\hat{\imath} = \hat{\imath}\, i$ etc. The complex quaternion may have imaginary parts of its parameters namely (a, A_1, A_2, A_3) such that

$$Q = (a + ib) + (A_1 + iB_1)\hat{\imath} + (A_2 + iB_2)\hat{\jmath} + (A_3 + iB_3)\hat{k} = a + ib + \vec{A} + i\vec{B} \qquad 3\text{-}1\text{-}1$$

Thus Q, a complex quaternion, may contain a real number, an imaginary number, a real vector, and an imaginary vector.

Now that complex quantities are present in the quaternion we must note that addition and subtraction are still well defined if the real and imaginary parts of the quaternion are grouped and acted on separately. Multiplication is also well defined. However we will now need three types of conjugation operations.

1) Complex Conjugation: changes the sign of i but leaves vectors alone. Denoted as Q^* (Q star).

$$Q^* = a - ib + \vec{A} - i\vec{B} \qquad 3\text{-}1\text{-}2$$

2) Hamiltonian Conjugation: leaves i alone but changes the sign of vectors and reverses order.

   Denoted as $Q^\times$ (Q cross).

$$Q^\times = a + ib - \vec{A} - i\vec{B} \qquad 3\text{-}1\text{-}3$$

2) Hermitean Conjugation: changes the sign of vectors and of i and also reverses order.

Effectively both Complex and Hamiltonian conjugation. Denote as Q^{T} (Q dagger).

$$Q^{\mathrm{T}} = \left(Q^{\times}\right)^{*} = \left(Q^{*}\right)^{\times} = a - ib - \vec{A} + i\vec{B} \qquad \text{3-1-4}$$

Table 3-1-1 summaries these effects on different forms of quaternions, q_1 q_2 and on a complex number, C.

Table 3-1-1: Effects of Conjugation

	*	×	T
$C = a + ib$	$a - ib$	$a + ib$	$a - ib$
$q_1 = a + \vec{A}$	$a + \vec{A}$	$a - \vec{A}$	$a - \vec{A}$
$q_2 = re^{\hat{n}\theta}$	$re^{\hat{n}\theta}$	$re^{-\hat{n}\theta}$	$re^{-\hat{n}\theta}$

NOTE: any two operations applied to a quaternion is equivalent to the third.

$$\begin{aligned} i.e. \left(q^{*}\right)^{\times} &= q^{\mathrm{T}} = \left(q^{\times}\right)^{*} \\ \left(q^{*}\right)^{\mathrm{T}} &= q^{\times} = \left(q^{\mathrm{T}}\right)^{*} \\ \left(q^{\times}\right)^{\mathrm{T}} &= q^{*} = \left(q^{\mathrm{T}}\right)^{\times} \end{aligned} \qquad \text{3-1-5}$$

3-2 Absolute Value of Complex Quaternions

The modulus of real quaternions was well defined previously using the Hamiltonian conjugation such that; given $q = a + \vec{A} = a + A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ as a real quaternion then

$$|q| = qq^{\times} = a^2 + A_1^2 + A_2^2 + A_3^2 \qquad \text{3-2-1}$$

Note that this holds under hermitean conjugation as $q^{\times} = q^{\mathrm{T}}$. Before considering the modulus of a complex quaternion, Q, we must note the restriction on Q if it is equal to one of its conjugates:

$$\begin{aligned} Q = Q^{\times} &\rightarrow Q = \frac{Q + Q^{\times}}{2} = \frac{(a + ib + \vec{A} + i\vec{B}) + (a + ib - \vec{A} - i\vec{B})}{2} \\ &= a + ib \\ Q = -Q^{\times} &\rightarrow Q = \vec{A} + i\vec{B} \end{aligned}$$

$$\begin{aligned} Q = Q^{*} &\rightarrow a + \vec{A} \\ Q = -Q^{*} &\rightarrow ib + i\vec{B} \end{aligned} \qquad \text{3-2-2}$$

$$\begin{aligned} Q = Q^{\mathrm{T}} &\rightarrow a + i\vec{B} \\ Q = -Q^{\mathrm{T}} &\rightarrow ib + \vec{A} \end{aligned}$$

Also the order in which the Hamiltonian and Hermitean conjugation are performed must be considered due to the reversal of factors:

$$\begin{aligned} (QQ^{\times})^{\times} &= Q^{\times\times}Q^{\times} = QQ^{\times} \\ (Q^{\times}Q)^{\times} &= Q^{\times}Q \end{aligned}$$

$$\text{3-2-3}$$

$$\begin{aligned} (QQ^{\mathrm{T}})^{\mathrm{T}} &= Q^{\mathrm{TT}}Q^{\mathrm{T}} = QQ^{\mathrm{T}} \\ (Q^{\mathrm{T}}Q)^{\mathrm{T}} &= Q^{\mathrm{T}}Q \end{aligned}$$

(note that the duals of each conjugation operation are not the same)

In trying to find the modulus of a complex quaternion it makes sense to follow the form for the real quaternion namely equation 3-2-1 or $\underset{\sim}{Q}\underset{\sim}{Q}^{\times}$ but from series 3-2-2 this results in a complex number. However we know that if we multiply a complex number by its complex conjugate a real number is formed:

$$\underset{\sim}{Q}\underset{\sim}{Q}^{\times}\left(\underset{\sim}{Q}\underset{\sim}{Q}^{\times}\right)^{*} = \underset{\sim}{Q}\underset{\sim}{Q}^{\times}\underset{\sim}{Q}^{*}\underset{\sim}{Q}^{\mathrm{T}} \qquad \text{3-2-4}$$

Since $\underset{\sim}{Q}\underset{\sim}{Q}^{\times} = \left|\underset{\sim}{Q}\right|^{2}$ and $\left(\underset{\sim}{Q}\underset{\sim}{Q}^{\times}\right)^{*} = \left(\left|\underset{\sim}{Q}\right|^{2}\right)^{*} = \left|\underset{\sim}{Q}\right|^{2}$ this expression will lead to

$\left|\underset{\sim}{Q}\right|^{4} = \underset{\sim}{Q}\underset{\sim}{Q}^{\times}\underset{\sim}{Q}^{*}\underset{\sim}{Q}^{\mathrm{T}}$ which is real and non-negative. Is it a modulus? This is answered if the expression of $\left|\underset{\sim}{Q}\right|^{4}$ is positive definite. Consider

$$\begin{aligned} \underset{\sim}{Q} &= 1 + i\hat{n} \\ \underset{\sim}{Q}^{\times} &= 1 - i\hat{n} \\ \underset{\sim}{Q}^{*} &= 1 - i\hat{n} \\ \underset{\sim}{Q}^{\mathrm{T}} &= 1 + i\hat{n} \end{aligned} \qquad \text{3-2-5}$$

Then $\underset{\sim}{Q}\underset{\sim}{Q}^{\times} = (1 + i\hat{n})(1 - i\hat{n}) = 1^{2} - (i\hat{n})^{2} = 1 - 1 = 0$ so that $\left|\underset{\sim}{Q}\right|^{4}$ is not positive definite. So $\left|\underset{\sim}{Q}\right| = \left[\underset{\sim}{Q}\underset{\sim}{Q}^{\times}\underset{\sim}{Q}^{*}\underset{\sim}{Q}^{\mathrm{T}}\right]^{1/4}$ cannot be a modulus. We have two types of complex quaternions:

1) Singular $\left|\underset{\sim}{Q}\right| = 0$

2) Non-Singular $\left|\underset{\sim}{Q}\right| \neq 0$

3-3 Exponential Form of Complex Quaternions

From real quaternions we saw that exponential forms can be found using Euler's formula. Again complex exponential forms may be found for complex quaternions except that a distinction must be made between singular and non-singular quaternions.

1) Non-Singular: $Q = re^{i\theta}e^{\hat{m}\phi}e^{i\hat{n}\alpha}$

$$
\begin{aligned}
&= re^{i\theta}e^{i\hat{n}'\alpha}e^{\hat{m}\phi} \\
&where\ \hat{n}' = e^{\hat{m}\phi}\hat{n}e^{-\hat{m}\phi} \\
&r = |Q| \neq 0
\end{aligned}
\qquad 3\text{-}3\text{-}1
$$

2) Singular: $Q = re^{i\theta}e^{\hat{m}\phi}\dfrac{1+i\hat{n}}{2}$

$$
\begin{aligned}
&= re^{i\theta}\frac{1+i\hat{n}'}{2}e^{\hat{m}\phi} \\
&where\ \hat{n}' = e^{\hat{m}\phi}\hat{n}e^{-\hat{m}\phi} \\
&r \neq |Q|
\end{aligned}
\qquad 3\text{-}3\text{-}2
$$

In both cases: $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{\hat{m}\phi} = \cos\phi + \hat{m}\sin\phi$

since $(i\hat{n})^2 = (i^2)(\hat{n}^2) = (-1)(-1) = 1$, we have:

$$
\begin{aligned}
e^{in} &= 1 + (i\hat{n})\alpha + (i\hat{n})^2\frac{\alpha^2}{2!} + (i\hat{n})^3\frac{\alpha^3}{3!} + \ldots \\
&= 1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \ldots + i\hat{n}\left(\alpha + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \ldots\right) \\
&= \cosh\alpha + i\hat{n}\sinh\alpha
\end{aligned}
\qquad 3\text{-}3\text{-}2
$$

3-4 The 4-Vector and 4 Co-Ordinates

Regular 4-vectors represented by the parameters (a_0, a_1, a_2, a_3) may be expressed in complex quaternion form as:

$$\underline{A} = A_0 + i\vec{A} \qquad \text{3-4-1}$$

where $\underline{A}$ (A "bar") is our notation for a 4-vector

A_0 is the scalar part

$\vec{A}$ is a real vector.

Thus an event given at (t, x, y, z) may be represented by a 4-vector as

$$\underline{x} = t + i\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = t + i\vec{r} \qquad \text{3-4-2}$$

An interval may also be represented by a 4-vector as $\underline{\Delta x} = \Delta t + i\left(\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}\right)$.

These 4-vectors as quaternions have a special property that we can use to our advantage later: they follow the form of 3-2-2 i.e. $\underline{A} = \underline{A}^{\mathrm{T}}$. Thus they are invariant under Hermitean conjugation.

3-5 Invariant Length of a 4-Vector

Look for the modulus of the 4-vector quaternion. It should be invariant under transformation between reference frames, since the 4-vector quaternion is Hermitean i.e.

$$\begin{aligned} \underline{AA}^{\times} &= \underline{A}^{\times}\underline{A} \\ &= A_0^2 - \left(i\vec{A}\right)^2 \\ &= A_0^2 - A_1^2 - A_2^2 - A_3^2 \end{aligned} \qquad \text{3-5-1}$$

We can now define three invariant transformation types:

1) $\underline{AA}^{\times} > 0$ $\underline{A}$ is time-like, magnitude $\sqrt{\underline{AA}^{\times}}$ 3-5-2a

2) $\underline{AA}^{\times} = 0$ $\underline{A}$ is light-like 3-5-2b

3) $\underline{AA}^{\times} < 0$ $\underline{A}$ is space-like, magnitude $\sqrt{-\underline{AA}^{\times}}$ 3-5-2c

A 4-vector representing a point in space-time, has the form $(t, x, y, z) = \underline{x} = t + i(x\hat{i} + y\hat{j} + z\hat{k})$ and we can form the invariant generating:

$$\underline{xx}^{\times} = (t + i\vec{r})(t - i\vec{r}) = t^2 + \vec{r}^2 = t^2 + (x\hat{i} + y\hat{j} + z\hat{k})^2 = t^2 - (x^2 + y^2 + z^2) \quad \text{3-5-3}$$

Similarly for an interval $\underline{\Delta x} = \Delta t + i\Delta\vec{r}$ between two points in space-time. We have

$$\underline{\Delta x \Delta x}^{\times} = (\Delta t + i\Delta\vec{r})(\Delta t - i\Delta\vec{r}) = \Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) \quad \text{3-5-4}$$

According to the classification in equation 3-5-2 the interval can be time-like, light-light, or space-like. However if we wish to discuss the velocity of a real particle; we should set the 4-vector invariant length as time-like. So

$$\vec{u} = \frac{\Delta \underline{x}}{\Delta \tau} \quad \text{3-5-5}$$

where $\Delta \underline{x} = \Delta t + i\Delta\vec{v}$

and $\Delta\tau = \sqrt{\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2} = \sqrt{\Delta \underline{x} \Delta \underline{x}^{\times}} > 0$

Thus we will deal with proper time intervals of invariant length of $\Delta\tau$.

3-6 Scalar Product of 4-Vectors

Given $\underline{A} = A_0 + i\vec{A}$ $\underline{B} = B_0 + i\vec{B}$ and noting $\underline{A}^* = A_0 - i\vec{A} = \underline{A}^\times$

Look at the modulus of $\underline{A}$ first:

$$|\underline{A}|^2 = A_0^2 - A_1^2 - A_2^2 - A_3^2 = A_0^2 - |\vec{A}|^2$$
$$= A_0^2 - \vec{A} \circ \vec{A} = A_0^2 + \vec{A}\vec{A} \qquad \text{3-6-1}$$

But $A_0^2 - |\vec{A}|^2 = (A_0 - i\vec{A})(A_0 + i\vec{A}) = \underline{A}^* \underline{A}$ and also $\underline{A}\underline{A}^* = \underline{A}^* \underline{A} = \underline{A}\underline{A}^\times = \underline{A}^\times \underline{A}$.

Thus the scalar product of $\underline{A}$ with itself is

$$|\underline{A}|^2 = \underline{A} \circ \underline{A} = \underline{A}\underline{A}^* = \underline{A}\underline{A}^\times \qquad \text{3-6-2}$$

which follows the known scalar result for normal 3-vectors. This is also true for $\underline{B}$.

Now let us explore the scalar product of two independent 4-vectors: $\underline{A} \circ \underline{B}$.

Define $(\underline{A} + \underline{B}) \circ (\underline{A} + \underline{B}) = \underline{A} \circ \underline{A} + \underline{B} \circ \underline{B} + 2\underline{A} \circ \underline{B}$

Since $(\underline{A} + \underline{B})(\underline{A}^* + \underline{B}^*) = \underline{A}\underline{A}^* + \underline{B}\underline{B}^* + \underline{A}\underline{B}^* + \underline{B}\underline{A}^*$, and $\underline{A}\underline{A}^* = \underline{A} \circ \underline{A}$ and similarly for $\underline{B}$,

we must have:

$$2\underline{A} \circ \underline{B} = \underline{A}\underline{B}^* + \underline{B}\underline{A}^* \qquad \text{3-6-3}$$

Thus we can now define the scalar product of two 4-vectors as:

$$\underline{A} \circ \underline{B} = \tfrac{1}{2}(\underline{A}\underline{B}^* + \underline{B}\underline{A}^*) \qquad \text{3-6-4}$$

or equivalently: $\underline{A} \circ \underline{B} = \tfrac{1}{2}(\underline{A}\underline{B}^\times + \underline{B}\underline{A}^\times) = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3$ 3-6-5

3-7 Lorentz Transformation of the Co-ordinates

In order to now deal with relativistic transformations of 4-vector co-ordinates, a form of the Lorentz transformation (L.T.) must be found. As shown in previous sections the length of the 4-vector must remain invariant under L.T. and also the resultant transformed 4-vector must remain Hermitean. This suggests that the required form is similar to the quaternion rotations form. Let L represent the L.T. operation which is a complex quaternion of unit modulus (non-singular). Thus the form is $L = e^{i\frac{\theta}{2}} e^{\hat{m}\frac{\theta}{2}} e^{i\hat{n}\frac{\alpha}{2}}$ and the L.T. of $\underline{A}$ is something like $\underline{A}' = L\underline{A}L^{\mathrm{T}}$ which is linear.

Now let's examine the operator L. First in L the term $e^{i\frac{\theta}{2}}$ will commute with $\underline{A}$ and cancels out $e^{-i\frac{\theta}{2}}$ in L^{T} and thus is redundant. The term $e^{\hat{m}\frac{\theta}{2}}$ is a rotation in 3-space which only acts on the imaginary vector part of $\underline{A}$. This effect is illustrated alone below:

$$\begin{aligned}
\underline{A}' &= e^{\hat{m}\frac{\theta}{2}} \underline{A} e^{-\hat{m}\frac{\theta}{2}} \\
&= e^{\hat{m}\frac{\theta}{2}} \left(A_0 + i\vec{A} \right) e^{-\hat{m}\frac{\theta}{2}} \\
&= e^{\hat{m}\frac{\theta}{2}} \left(A_0 \right) e^{-\hat{m}\frac{\theta}{2}} + e^{\hat{m}\frac{\theta}{2}} \left(i\vec{A} \right) e^{-\hat{m}\frac{\theta}{2}} \\
&= A_0 e^{\hat{m}\frac{\theta}{2}} e^{-\hat{m}\frac{\theta}{2}} + i e^{\hat{m}\frac{\theta}{2}} \vec{A} e^{-\hat{m}\frac{\theta}{2}} \\
&= A_0 + i e^{\hat{m}\frac{\theta}{2}} \vec{A} e^{-\hat{m}\frac{\theta}{2}}
\end{aligned} \qquad \text{3-7-1}$$

which only rotates $\vec{A}$ through θ about an axis of direction $\hat{m}$.

This operation has been covered previously and is not important now in discussing the L.T. Now only the term $e^{i\hat{n}\frac{\alpha}{2}}$ is left. This will be known as the Lorentz Boost. Its effect will be to relativistically add the velocity $\vec{v}$ to the system as a whole (co-ordinates, velocity, and momentum) or if you will "boost" the system to a new value. This boost can be viewed under two cases as were the rotations:

Case 1) Active Boost: The system consisting of objects with velocity, co-ordinates, and momentum has a velocity $\vec{v}$ relativistically added.

$$\text{i.e. } \underline{A}' = e^{+i\hat{v}\frac{\alpha}{2}}\underline{A}e^{+i\hat{v}\frac{\alpha\theta}{2}} \qquad 3\text{-}7\text{-}2$$

Case 2) Passive Boost: The system is transformed to a new reference frame traveling with velocity $\vec{v}$ relative to the old frame and all measurements are thus measured from the initial frame.

$$\text{i.e. } \underline{A}' = e^{-i\hat{v}\frac{\alpha}{2}}\underline{A}e^{-i\hat{v}\frac{\alpha\theta}{2}} \qquad 3\text{-}7\text{-}3$$

Thus the L.T. is a product of two terms representing a rotation and the Lorentz boost (relativistic shift).

However, so far we have not discussed the form of 4-velocity and 4-momentum; but have just assumed that they are time-like. This is to say that they act similarly to 3-space velocity and momentum. We have in fact assumed that they are time derivatives of the co-ordinates. The next section will clarify this for the time-like condition for 4-velcotiy (we have in fact just "cornered" our options on 4-velocity and 4-momentum). Recall in that the time-like condition was chosen in section 3-4 when defining the 4-vector so we are correct in assuming the time-like condition.

3-8 4-Velocity, Intervals, and Rapidity (α) in Space-Time

If we set an object moving in space-time with a velocity then the interval between two positions (events) is given by:

$$\underline{\Delta x} = \Delta t + i\Delta\vec{r} \qquad 3\text{-}8\text{-}1$$

The form of velocity in Newtonian-Euclidean geometry is $\vec{v} = \dfrac{\Delta\vec{r}}{\Delta t}$ where Δt is invariant. Then in 4-space we would need: $\underline{u} = \dfrac{\Delta x}{\Delta\tau}$ where Δτ is the invariant time-like length of the 4-vector $\underline{\Delta x}$ (same in all frames). Thus

$$\underline{u} = \frac{\Delta t}{\Delta\tau} + i\left(\frac{\Delta x}{\Delta\tau}\hat{i} + \frac{\Delta y}{\Delta\tau}\hat{j} + \frac{\Delta z}{\Delta\tau}\hat{k}\right) = (u_0, u_1, u_2, u_3) \qquad \text{3-8-2}$$

where $\Delta\tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

We can now use space-time relationships to define the rapidity α of the object as

$$\alpha = \tan^{-1} v \qquad \text{3-8-3}$$

This is the angle of the Minkowski triangle given in hyperbolic trigonometry

i.e. $\Delta t = \Delta\tau \cosh\alpha$

$|\Delta\vec{r}| = \Delta\tau \sinh\alpha$

$v = \Delta\vec{r} / \Delta t = \tanh\alpha$

Limits $\alpha \geq 0$; $0 \leq \tanh\alpha < 1$

If we also define $\gamma = \left(1 - v^2\right)^{-\frac{1}{2}} = \cosh\alpha$ 3-8-4

Then $\underline{u} = u_0 + i\vec{u} = \gamma + \gamma i\vec{v} = \dfrac{\Delta t}{\Delta\tau} + i\dfrac{|\Delta\vec{r}|}{\Delta\tau}$ 3-8-5

Also note that $u_0^2 - u_1^2 - u_2^2 - u_3^2 = 1$. Thus every 4-velocity of a real object is a unit time-like 4-vector.

A special case: Photons

Consider the interval $\underline{\Delta x}$ between two events on the world line of a photon ($\Delta\tau = 0$).

$$\frac{|\Delta\vec{r}|}{\Delta t} = c = 1$$
$$\therefore\ \Delta\tau^2 = \Delta t^2 - |\Delta\vec{r}|^2 = 0 \qquad \text{3-8-6}$$

so $u_o = \dfrac{\Delta t}{\Delta\tau}$, $u_1 = \dfrac{\Delta x}{\Delta\tau}$ etc. are undefined. This is the limiting case where $\alpha \to \infty$ yielding the speed of light.

As with all unit 4-vectors; 4-velocity has an unimodular form. We can find this by recognizing:

$$\Delta t = \Delta\tau \cosh\alpha$$
$$\Delta\vec{r} = \Delta\tau \sinh\alpha\,\hat{n} \qquad \text{3-8-7}$$

where $\vec{r}$ is the position vector.

$$\begin{aligned}\Delta\underline{x} &= \Delta t + i\Delta\vec{r} = \Delta t + i\hat{n}\Delta r\\ &= \Delta\tau(\cosh\alpha + i\hat{n}\sinh\alpha)\\ &= \Delta\tau\, e^{i\hat{n}\alpha}\end{aligned} \qquad \text{3-8-8}$$

And thus the 4-velocity can be found as $\underline{u} = \dfrac{\Delta\underline{x}}{\Delta\tau} = e^{i\hat{n}\alpha}$ 3-8-9

which is a nice simple form.

3-9 Lorentz Transformation of 4-Velcoity

We had for 4-coordinates a form of the passive Lorentz boost: $\underline{x}' = e^{i\hat{v}\frac{\alpha}{2}}\underline{x}e^{-i\hat{v}\frac{\alpha\theta}{2}}$. This holds equally well for an interval $\Delta\underline{x}'$. We can now use the definition of 4-velocity using the invariant length to find the form of the L.T.

$$\begin{aligned} \Delta\underline{x}' &= e^{i\hat{v}\frac{\alpha}{2}}\Delta\underline{x}e^{i\hat{v}\frac{\alpha\theta}{2}} \\ \frac{\Delta\underline{x}'}{\Delta\tau} &= e^{i\hat{v}\frac{\alpha}{2}}\frac{\Delta\underline{x}}{\Delta\tau}e^{i\hat{v}\frac{\alpha\theta}{2}} \\ \underline{u}' &= e^{i\hat{v}\frac{\alpha}{2}}\underline{u}e^{i\hat{v}\frac{\alpha\theta}{2}} \end{aligned} \qquad \text{3-9-1}$$

where $\underline{u}$ is given as $\underline{u} = u_0 + i\left(u_1\hat{i} + u_2\hat{j} + u_3\hat{k}\right)$.

This is the same form as for the co-ordinates which is not surprising as we laboured to set the 4-velocity to be time invariant. Thus the rationale to set $\Delta\tau$ invariant is now clear.

This example will illustrate the nature of the previously derived 4-vector L.T. we will see that it can be viewed as a passive "rotation" in space time. Now we will use matrix method of standard L.T. to derive the quaternion form.

Let a primed frame be moving at v_0 in the x direction relative to the inertial, unprimed frame represent the transformation; under the L.T., of an object's world line AB.

Then $\gamma = \cosh\alpha = \frac{1}{\sqrt{1-v^2}}$ and $\Delta t' = \gamma_0 t - \gamma_0 \Delta x$.

In order to simplify the algebra we can express each individual element's transformation in a 4 element column matrix:

$$\begin{bmatrix}\Delta t'\\ \Delta x'\\ \Delta y'\\ \Delta z'\end{bmatrix}=\begin{bmatrix}\gamma_0 & -\gamma_0 v_0 & 0 & 0\\ -\gamma_0 v_0 & \gamma_0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\Delta t\\ \Delta x\\ \Delta y\\ \Delta z\end{bmatrix} \qquad \text{3-9-2}$$

For now we can omit $\Delta y'$, $\Delta z'$ as they remain unchanged.

Thus $$\begin{bmatrix}\Delta t'\\ \Delta x'\end{bmatrix}=\gamma_0\begin{bmatrix}1 & -v_0\\ -v_0 & 1\end{bmatrix}\begin{bmatrix}\Delta t\\ \Delta x\end{bmatrix} \qquad \text{3-9-3}$$

However we know from the Minkowski Triangle in the space-time diagram that $\gamma_0 = \cosh\alpha$ and $v_0 = \tanh\alpha$ (see section 3-8)

Thus $\begin{bmatrix}\Delta t'\\ \Delta x'\end{bmatrix}=\begin{bmatrix}\cosh\alpha & -\sinh\alpha\\ -\sinh\alpha & \cosh\alpha\end{bmatrix}\begin{bmatrix}\Delta t\\ \Delta x\end{bmatrix}$ we recall that $e^{i\hat{n}\alpha} = \cosh\alpha + i\hat{n}\sinh\alpha$ and we can combine this 2 X 2 matrix into a linear equation:

$$\begin{aligned}\Delta t' &= \Delta t\cosh\alpha - \Delta x\sinh\alpha\\ \Delta x' &= \Delta x\cosh\alpha - \Delta t\sinh\alpha\end{aligned} \qquad \text{3-9-4}$$

Now substitute for $\Delta t'$ and $\Delta x'$ from 3-9-4 and we find

$$\begin{aligned}\Delta t' + i\hat{n}\Delta x' &= \Delta t\cosh\alpha - \Delta x\sinh\alpha + i\hat{n}(\Delta x\cosh\alpha - \Delta t\sinh\alpha)\\ &= (\Delta t + i\hat{n}\Delta x)\cosh\alpha - (i\hat{n}\Delta t + \Delta x)\sinh\alpha\\ &= (\Delta t + i\hat{n}\Delta x)\cosh\alpha - i\hat{n}\left((i\hat{n})^2\Delta t + i\hat{n}\Delta x\right)\sinh\alpha\\ &= (\cosh\alpha - \sinh\alpha)(\Delta t + i\hat{n}\Delta x) \qquad \text{3-9-5}\\ &= e^{-i\hat{n}\alpha}(\Delta t + i\hat{n}\Delta x)\\ &= e^{-i\hat{n}\frac{\alpha}{2}}e^{-i\hat{n}\frac{\alpha}{2}}(\Delta t + i\hat{n}\Delta x)\\ &= e^{-i\hat{n}\frac{\alpha}{2}}(\Delta t + i\hat{n}\Delta x)e^{-i\hat{n}\frac{\alpha}{2}}\end{aligned}$$

noting that $\hat{n}^2 = -1$, $i^2 = -1$ so $(i\hat{n})^2 = 1$

Does this form hold for $\Delta y', \Delta z'$ which we omitted? Consider this:

$$\begin{aligned} i\left(\hat{j}\Delta y' + \hat{k}\Delta z'\right) &= i\left(\hat{j}\Delta y + \hat{k}\Delta z\right) \\ &= e^{-i\hat{n}\frac{\alpha}{2}} e^{i\hat{n}\frac{\alpha}{2}} i\left(\hat{j}\Delta y + \hat{k}\Delta z\right) \\ &= e^{-i\hat{n}\frac{\alpha}{2}} i e^{i\hat{n}\frac{\alpha}{2}} \left(\hat{j}\Delta y + \hat{k}\Delta z\right) \\ &= e^{-i\hat{n}\frac{\alpha}{2}} i\left(\hat{j}\Delta y + \hat{k}\Delta z\right) e^{-i\hat{n}\frac{\alpha}{2}} \end{aligned} \qquad 3\text{-}9\text{-}6$$

Thus if we let $\Delta \underline{r}' = \Delta t' + i\left(\hat{n}\Delta x + \hat{j}\Delta y + \hat{k}\Delta z\right)$ we have a L.T. of the form

$\Delta \underline{r}' = e^{-i\hat{n}\frac{\alpha}{2}} \Delta \underline{r}\, e^{-i\hat{n}\frac{\alpha}{2}}$ which is the same form previously derived and is indeed correct.

3-10 Examples of the Lorentz Transformation

Example 1 Frame O' is moving at rapidity $\alpha = \ln(3)$ in the positive z direction relative to frame O. In frame O, event A occurs at (t,x,y,z) = (5,2,3,4); also, a rocket is moving at rapidity $\beta = \ln(2)$ in the y direction. Find the co-ordinates of event A and the 4-velcoity of the rocket in frame O'.

Solution: In frame O we know $\underline{A} = (5,2,3,4) \Rightarrow \underline{x} = 5 + i\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)$ 3-10-1

apply transformation: $e^{-i\hat{k}\frac{\alpha}{2}} = \frac{1}{\sqrt{3}}\left(2 - i\hat{k}\right)$ 3-10-2

since

$$\begin{aligned} \cosh\frac{1}{2}\ln 3 &= \frac{2}{\sqrt{3}} \\ \sinh\frac{1}{2}\ln 3 &= \frac{1}{\sqrt{3}} \end{aligned} \qquad 3\text{-}10\text{-}3$$

Then

$$\begin{aligned}
\underline{x}' &= e^{-i\hat{k}\frac{\alpha}{2}}\left(5+i\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\right)e^{-i\hat{k}\frac{\alpha}{2}} \\
&= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\left(2-i\hat{k}\right)\left(5+i\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\right)\left(2-i\hat{k}\right) \\
&= \frac{1}{3}\left(2-i\hat{k}\right)\left(10+i\left(4\hat{i}+6\hat{j}+8\hat{k}\right)\right)-5i\hat{k}-i^2\left(2\left(-\hat{j}\right)+3\hat{i}+4(-1)\right) \\
&= \frac{1}{3}\left(2-i\hat{k}\right)\left(6+i\left(4\hat{i}+6\hat{j}+3\hat{k}\right)+3\hat{i}-2\hat{j}\right) \\
&= \frac{1}{3}\left(12+2i\left(4\hat{i}+6\hat{j}+3\hat{k}\right)+6\hat{i}-4\hat{j}-6i\hat{k}-i^2\left(4\hat{j}-6\hat{i}+3(-1)\right)-i\left(3\hat{j}-2\left(-\hat{i}\right)\right)\right) \\
&= \frac{1}{3}\left(9+i\left(6\hat{i}+9\hat{j}\right)\right) \\
&= 3+i\left(2\hat{i}+3\hat{j}\right)
\end{aligned} \qquad \text{3-10-4}$$

in O: $\underline{u} = \dfrac{\Delta\underline{x}}{\Delta\tau} = e^{i\hat{j}\beta} = \cosh(\ln 2) + i\hat{j}\sinh(\ln 2) = \dfrac{1}{4}\left(5+3i\hat{j}\right)$ 3-10-5

in O':

$$\begin{aligned}
\underline{u}' &= \frac{\Delta\underline{x}'}{\Delta\tau} = e^{-i\hat{k}\frac{\alpha}{2}}\underline{u}e^{-i\hat{k}\frac{\alpha}{2}} \\
&= \frac{1}{12}\left(2-i\hat{k}\right)\left(5+3i\hat{j}\right)\left(2-i\hat{k}\right) \\
&= \frac{1}{12}\left(2-i\hat{k}\right)\left(10-5i\hat{k}+6i\hat{j}-3i^2\hat{i}\right) \\
&= \frac{1}{12}\left(20+2i\left(6\hat{j}-5\hat{k}\right)+6\hat{i}-10i\hat{k}-i^2\left(-\hat{i}6-5(-1)\right)-3i\hat{j}\right) \\
&= \frac{1}{12}\left(20+i\left(12\hat{j}-20\hat{k}\right)+6\hat{i}-6\hat{i}+5-3i\hat{j}\right) \\
&= \frac{1}{12}\left(25+i\left(9\hat{j}-20\hat{k}\right)\right)
\end{aligned} \qquad \text{3-10-6}$$

3-11 4-Momentum

We expect the 4-momentum to transform like a 4-vector, so we can follow the 3-space definition

$$\vec{P} = m\vec{v} \qquad \text{3-11-1}$$

Thus $\underline{P} = E + i\vec{P} = m\underline{u} = me^{i\hat{v}\alpha}$ where m is the mass of the particle. 3-11-2

Then also energy: E = m cosh(α) and momentum: P = m sinh(α). Again 4-momentum is time-like but no longer a unit 4-vector:

$$\underline{P}\underline{P}^{\times} = me^{i\hat{v}\alpha}me^{-i\hat{v}\alpha} = m^2 = \left(E + i\vec{P}\right)\left(E - i\vec{P}\right) = E^2 - \left|\vec{P}\right|^2 \qquad \text{3-11-3}$$

For a zero mass particle we will not have a 4-velocity $\underline{u}$, but we can define

$$\begin{aligned} &\underline{P} = E + i\vec{P} \\ &\underline{P}\underline{P}^{\times} = E^2 - \left|\vec{P}\right|^2 = 0 \\ &E = \left|\vec{P}\right| \end{aligned} \qquad \text{3-11-4}$$

which is expected.

3-12 L.T. of 4-Momentum

Again since 4-momentum is a 4-vector we can find the required L.T. by directly following the definition of momentum and using the known L.T. of 4-velocity. Then

$$\begin{aligned} \underline{P}' &= m\underline{u}' = me^{-i\hat{n}\frac{\alpha}{2}}\underline{u}e^{-i\hat{n}\frac{\alpha}{2}} \\ &= e^{-i\hat{n}\frac{\alpha}{2}}m\underline{u}e^{-i\hat{n}\frac{\alpha}{2}} \\ &= e^{-i\hat{n}\frac{\alpha}{2}}\underline{P}e^{-i\hat{n}\frac{\alpha}{2}} \end{aligned} \qquad \text{3-12-1}$$

as expected.

3-13 Invariance of the Hermitean Property, Scalar Product, and Proper Intervals under the Lorentz Transformation

Recall in sections 3-4 and 3-7, these properties of the 4-vectors and of the Lorentz transformation acting on them:

$$\begin{aligned}
&\underline{A} = \underline{A}^{\mathrm{T}} \\
&\underline{A}^{\times} = \underline{A}^{*} \\
&\underline{A}' = L\underline{A}L^{\mathrm{T}} \\
&\left(\underline{A}'\right)^{\times} = \left(L\underline{A}L^{\mathrm{T}}\right)^{\times} = L^{*}\underline{A}^{\times}L^{\times} \\
&\left(\underline{A}'\right)^{*} = L^{*}\underline{A}^{*}L^{\times}
\end{aligned} \qquad \text{3-13-1}$$

Is this Hermitean property conserved under the L.T.? i.e. is it true that $\underline{A}' = \left(\underline{A}'\right)^{\mathrm{T}}$

$$\begin{aligned}
\left(\underline{A}'\right)^{\mathrm{T}} &= \left(L\underline{A}L^{\mathrm{T}}\right)^{\mathrm{T}} \\
&= \left(L^{\mathrm{T}}\right)^{\mathrm{T}}\underline{A}^{\mathrm{T}}L^{\mathrm{T}} \\
&= L\underline{A}^{\mathrm{T}}L^{\mathrm{T}} \\
&= L\underline{A}L^{\mathrm{T}} \\
&= \underline{A}'
\end{aligned} \qquad \text{3-13-2}$$

not surprisingly it is conserved.

Recall in section 3-6 the scalar product of 4-vectors was defined as

$$\underline{A} \circ \underline{B} = \tfrac{1}{2}\left(\underline{A}\underline{B}^{*} + \underline{B}\underline{A}^{*}\right) = \tfrac{1}{2}\left(\underline{A}\underline{B}^{\times} + \underline{B}\underline{A}^{\times}\right) \qquad \text{3-13-3}$$

Now is this invariant under the L.T.? Consider

$$
\begin{aligned}
(\underline{A}\circ\underline{B})' &= \underline{A}'\circ\underline{B}' \\
&= \tfrac{1}{2}\left[\underline{A}'\left(\underline{B}'\right)^{\times}+\underline{B}'\left(\underline{A}'\right)^{\times}\right] \\
&= \tfrac{1}{2}\left[L\underline{A}L^{\mathrm{T}}\left(L\underline{B}L^{\mathrm{T}}\right)^{\times}+L\underline{B}L^{\mathrm{T}}\left(L\underline{A}L^{\mathrm{T}}\right)^{\times}\right] \\
&= \tfrac{1}{2}\left[L\underline{A}L^{\mathrm{T}}\left(L^{*}\underline{B}L^{\times}\right)+L\underline{B}L^{\mathrm{T}}\left(L^{*}\underline{A}L^{\times}\right)\right] \\
&= \tfrac{1}{2}\left[L\underline{A}L^{\mathrm{T}}L^{*}\underline{B}L^{\times}+L\underline{B}L^{\mathrm{T}}L^{*}\underline{A}L^{\times}\right] \qquad \text{3-13-4} \\
&= \tfrac{1}{2}\left[L\underline{A}\underline{B}L^{\times}+L\underline{B}\underline{A}L^{\times}\right] \\
&= \tfrac{1}{2}L\left[\underline{A}\underline{B}+\underline{B}\underline{A}\right]L^{\times} \\
&= \tfrac{1}{2}LL^{\times}\left[\underline{A}\underline{B}+\underline{B}\underline{A}\right] \\
&= \tfrac{1}{2}\left[\underline{A}\underline{B}+\underline{B}\underline{A}\right] \\
&= \underline{A}\circ\underline{B}
\end{aligned}
$$

So we see that the scalar product of 4-vector is indeed Lorentz invariant.

3-14 Equation of Motion and the "4-Force"

Again 3-space results can be used as a guideline for finding the 4-vector equivalent. From Newton's second law we know: $\dfrac{d\vec{P}}{dt}=\vec{F}$ 3-14-1

Thus the direct form for 4-vectors should be like $\dfrac{d\underline{P}}{d\tau}=\underline{f}$ assuming that mass is constant (that is not undergoing chemical change, undergoing an inelastic collision, or radioactive decay).

The question is what does $\underline{f}$ look like? It must act on a particle causing it to travel on a trajectory. This is to say we must be able to solve for the new momentum of the particle, $\underline{P}(\tau+d\tau)$ from an initial momentum, $\underline{P}(\tau)$ over a proper time interval. We know that this change must be due to the Lorentz Transformation of the 4-momentum.

Recall that we had:

$$\underline{P}' = e^{i\hat{n}\frac{\alpha}{2}} \underline{P} e^{i\hat{n}\frac{\alpha}{2}} = e^{\hat{M}\frac{\theta}{2}} e^{i\hat{n}\frac{\alpha}{2}} \underline{P} e^{i\hat{n}\frac{\alpha}{2}} e^{\hat{M}\frac{\theta}{2}}$$
$$= L\underline{P}L^{\mathrm{T}} = \exp\left(\vec{L}\right)\underline{P}\exp\left(\vec{L}^{\mathrm{T}}\right) \qquad \text{3-14-2}$$

where $\vec{L}$ is a complex vector.

So let us effect this transformation on the new momentum:

$$\underline{P} \rightarrow \underline{P}(\tau + d\tau) = L(d\tau)\underline{P}L^{\mathrm{T}}(d\tau)$$
$$= \exp\left(\vec{L}(d\tau)\right)\underline{P}\exp\left(\vec{L}^{\mathrm{T}}(d\tau)\right)$$
$$= \exp\left(\vec{M}d\tau\right)\underline{P}\exp\left(\vec{M}^{\mathrm{T}}d\tau\right)$$
$$= \left(1 + \vec{M}d\tau\right)\underline{P}\left(1 + \vec{M}^{\mathrm{T}}d\tau\right) \qquad \text{3-14-3}$$
$$= \underline{P} + \underline{P}\vec{M}^{\mathrm{T}}d\tau + \vec{M}d\tau\underline{P} + \vec{M}\underline{P}\vec{M}^{\mathrm{T}}d\tau^2$$
$$= \underline{P} + \left(\vec{M}\underline{P} + \underline{P}\vec{M}^{\mathrm{T}}\right)d\tau + 0\left(d\tau^2\right)$$

where $\vec{M}$ is an arbitrary complex vector.

$$so \quad \underline{P}(\tau + d\tau) = \underline{P} + d\underline{P} = \underline{P} + \left(\vec{M}\underline{P} + \underline{P}\vec{M}^{\mathrm{T}}\right)d\tau$$
$$\therefore \frac{d\underline{P}}{d\tau} = \vec{M}\underline{P} + \underline{P}\vec{M}^{\mathrm{T}} = f \qquad \text{3-14-4}$$

Which is consistent with Newton's second law. Thus we have the equation of motion in quaternion form for a particle.

3-15 Lorentz Transformation of the 4-Force

In order to find the L.T. of this 4-force we need to recall that dτ is invariant under such a transformation. With this knowledge we can easily act the L.T. on this 4-force:

$$\begin{aligned} L\frac{d\underline{P}}{d\tau}L^{\mathrm{T}} &= L\left(\vec{M}\underline{P}+\underline{P}\vec{M}^{\mathrm{T}}\right)L^{\mathrm{T}} \\ &= L\vec{M}\underline{P}L^{\mathrm{T}} + L\underline{P}\vec{M}^{\mathrm{T}}L^{\mathrm{T}} \\ &= L\vec{M}L^{\times}L\underline{P}L^{\mathrm{T}} + L\underline{P}L^{\mathrm{T}}L^{*}\vec{M}^{\mathrm{T}}L^{\mathrm{T}} \\ &\therefore \vec{M}' = L\vec{M}L^{\times}, \\ \vec{M}^{\mathrm{T}'} &= \left(L\vec{M}L^{\times}\right)^{\mathrm{T}} = L^{*}\vec{M}^{\mathrm{T}}L^{\mathrm{T}} \end{aligned} \qquad 3\text{-}15\text{-}1$$

We now have both the equation of motion and the L.T. for the 4-force.

4 Electromagnetism: Microscopic Theory

4-1 Unit Considerations

Before progressing on further to special relativity or EM theory, we must look at units for space-time. For the time being, for simplicity, set c = 1. We have three common unit systems to choose from:

Table 4-1-1: Systems of Units

	Heavyside Lorentz	Guassian	MKS (rationalized)
ϵ_0	1	1	$10^7/4\pi c^2$
μ_0	1	1	$4\pi/10^7$
D	E+P	E+4πP	ϵ_0E+P
H	P-M	b-4πM	$(1/\mu_0)$B-M

In order to simplify the algebra as much as possible the Heavyside Lorentz system will be used (although any system of units is valid).

4-2 Vector Identities

Before we look at electromagnetism we need to have quaternion forms of partial derivatives, and some identities. First look at quaternion calculus:

Recall in section 2-1 we had written $\vec{A}\vec{B} = -\vec{A} \circ \vec{B} + \vec{A} \times \vec{B}$ 4-2-1 we can then conclude:

$$\tfrac{1}{2}(\vec{A}\vec{B}+\vec{B}\vec{A}) = -\vec{A}\circ\vec{B}$$
$$\tfrac{1}{2}(\vec{A}\vec{B}-\vec{B}\vec{A}) = \vec{A}\times\vec{B} \qquad 4\text{-}2\text{-}2$$

Now let us define the "Del" operator: $\nabla \equiv \hat{i}\partial_x + \hat{j}\partial_y + \hat{k}\partial_z$

Thus

$$\nabla\phi = grad\,\phi$$
$$\nabla\vec{A} = -div\,\vec{A} + curl\,\vec{A} = -\nabla\circ\vec{A} + \nabla\times\vec{A} \qquad 4\text{-}2\text{-}3$$

The Laplacian: $\nabla^2 = \nabla\nabla = -\left(\partial_x^{\,2} + \partial_y^{\,2} + \partial_z^{\,2}\right) \equiv -\Delta$ 4-2-4

Note the negative sign here. So the gradient acts like a vector and the Laplacian as a scalar. Similarly many other identities in three co-ordinates are also valid.

4-3 The Differential Field Operator

The scalar field operator should be invariant. Given ϕ, a scalar field then $d\phi$ should be the result of a scalar product of two 4-vectors. Let us try to decompose them:

$$\begin{aligned} d\phi &= \partial_t\phi\,dt + \partial_x\phi\,dx + \partial_y\phi\,dy + \partial_z\phi\,dz \\ &= \left[\partial_t\phi - i\left(\partial_x\phi\hat{i} + \partial_y\phi\hat{j} + \partial_z\phi\hat{k}\right)\right]\circ\left[dt + i\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)\right] \\ &= \left[(\partial_t - i\nabla)\phi\right]\circ\left[d\underline{x}\right] \\ &= D\phi\circ d\underline{x} \end{aligned} \qquad 4\text{-}3\text{-}1$$

Thus the 4-vector field operator or 4-gradient; is $\boxed{D = \partial_t - i\nabla}$ 4-3-2

Now look for the Laplacian of this operator. Consider $D^* = \partial_t + i\nabla$ 4-3-3

Then

$$\begin{aligned} DD^* &= (\partial_t - i\nabla)(\partial_t + i\nabla) = D^*D \\ &= \partial_t^2 + \nabla^2 \\ &= \partial_t^2 - \Delta \\ &= \partial_t^2 - \partial x^2 - \partial y^2 - \partial z^2 \\ &= \Box \end{aligned} \qquad 4\text{-}3\text{-}4$$

This is the D'Alembertian operator.

4-4 Conservation of Charge: 4-Current

If we wish to look for 4-vector forms of Maxwell's equations we need to consider charge. We expect charge to be conserved. Recall from vectorial Electromagnetism:

charge density ρ from $Q = \int_V \rho dV$ 4-4-1

current density $\vec{J}$ from $I = \int_S \vec{J} \circ \hat{n} dS$ 4-4-2

The 4-vector form will allow us to combine these scalar and vector quantities into a "4-current". Namely: $\underline{J} = \rho + i\vec{J}$ 4-4-3

For charge conservation to be true mathematically

$$\frac{\partial \rho}{\partial t} + \nabla \circ \vec{J} = 0 \Leftrightarrow D \circ \underline{J} = 0 \Leftrightarrow (\partial_t - i\nabla) \circ (\rho + i\vec{J}) = 0 \qquad 4\text{-}4\text{-}4$$

which is an equivalent form for a 4-vector!

4-5 Maxwell Equations in Quaternion Form

We have four known results:

$$\nabla \circ \vec{E} = \rho$$
$$\nabla \circ \vec{B} = 0$$
$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \qquad \text{4-5-1}$$
$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

where these are the microscopic (Lorentz) forms of Maxwell's equations.

First we need to define the E&M field as $F = -\vec{B} + i\vec{E} = i\left(\vec{E} + i\vec{B}\right)$. 4-5-2

We expect that Maxwell's equations will be condensed as the 4-vectors carry the scalar quantities. Since D is operating similarly to Del let us make the claim that:

$$DF^{*} = \underline{J} \Leftrightarrow D^{*}F = \underline{J}^{*} \qquad \text{4-5-3}$$

To prove this use Maxwell's equations and the identity $\nabla\vec{V} = -\nabla \circ \vec{V} + \nabla \times \vec{V}$ 4-5-4

Proof:

$$\begin{aligned} DF^{*} &= \left(\partial_t - i\nabla\right)\left(-\vec{B} - i\vec{E}\right) \\ &= -\partial_t\vec{B} - i\partial_t\vec{E} + i\nabla\vec{B} - \nabla\vec{E} \\ &= -\partial_t\vec{B} - i\partial_t\vec{E} + i\left(-\nabla \circ \vec{B} + \nabla \times \vec{B}\right) - \left(-\nabla \circ \vec{E} + \nabla \times \vec{E}\right) \\ &= \left(-\partial_t\vec{B} + \nabla \circ \vec{E} - \nabla \times \vec{E}\right) + i\left(-\partial_t\vec{E} - \nabla \circ \vec{B} + \nabla \times \vec{B}\right) \\ &= \nabla \circ \vec{E} - \left(\nabla \times \vec{E} + \partial_t\vec{B}\right) - i\left(\nabla \circ \vec{B}\right) + i\left(\nabla \times \vec{B} - \partial_t\vec{E}\right) \\ &= \rho - (0) - i(0) + i\vec{J} \\ &= \underline{J} \end{aligned} \qquad \text{4-5-5}$$

This is the quaternion form of Maxwell's equation (it now makes sense to use singular tense).

We should also be able to derive the field from the potential. First define the quaternion potential as $\underline{A} = (\phi + i\vec{A})$ which is consistent with $\vec{B} = \nabla \times \vec{A}$ and

$$\begin{aligned}
&\nabla \times \vec{E} + \nabla \times \partial_t \vec{A} = 0 \\
&\nabla \times (\vec{E} + \partial_t \vec{A}) = 0 \\
&\vec{E} + \partial_t \vec{A} = -\nabla \phi
\end{aligned} \qquad \text{4-5-6}$$

Then

$$\begin{aligned}
F &= -\vec{B} + i\vec{E} \\
&= -\nabla \times \vec{A} - i\left(\nabla \Phi + \frac{\partial \vec{A}}{\partial t}\right) \\
&= (\partial_t - i\nabla)(\Phi - i\vec{A}) - (\partial_t \Phi + \nabla \vec{A}) \\
&= D\underline{A}^* - D \circ \underline{A}
\end{aligned} \qquad \text{4-5-7}$$

If we impose the Lorentz condition: $D \circ \underline{A} = 0$ then

$$\begin{aligned}
&D \circ \underline{A} = 0 \\
&\quad 0 = (\partial_t - i\nabla) \circ (\phi - i\vec{A}) \\
&\quad 0 = \partial_t \phi + \nabla \circ \vec{A}
\end{aligned} \qquad \text{4-5-8}$$

So we have $F = D\underline{A}^* - D \circ \underline{A} = D\underline{A}^*$ 4-5-9

We can now easily derive the field from the potential:

$$
\begin{aligned}
F &= D\underline{A}^* \\
&= (\partial_t - i\nabla)(\phi - i\vec{A}) \\
&= \partial_t\phi - i\partial_t\vec{A} - i\nabla\phi - \nabla\vec{A} \\
&= \partial_t\phi - \nabla\vec{A} - i(\partial_t\vec{A} + \nabla\phi) \\
&= \partial_t\phi + \nabla\circ\vec{A} - \nabla\times\vec{A} + i(-\nabla\phi - \partial_t\vec{A}) \\
&= (\partial_t\phi + \nabla\circ\vec{A}) - \vec{B} + i\vec{E} \\
&= D\circ\vec{A} + F \\
&= F
\end{aligned}
\qquad \text{4-5-10}
$$

4-6 4-Potential Gauge Transformation

The gauge transformation of the 4-potential will be most useful as it is independent of the co-ordinate system used. The transformation is accomplished by (recall c = 1).

$$
\begin{aligned}
\vec{A}' &= \vec{A} + \nabla\Lambda \\
\phi' &= \phi - \frac{\partial\Lambda}{\partial t}
\end{aligned}
\qquad \text{4-6-1}
$$

combine this as a 4-vector: $\phi' + i\vec{A}' = \phi + i\vec{A} - \left(\frac{\partial\Lambda}{\partial t} - i\nabla\Lambda\right)$ 4-6-2

So $\underline{A}' = \underline{A} - D\Lambda$ or $\underline{A}'^* = \underline{A}^* - D^*\Lambda$ 4-6-3

Under a gauge transformation:

$$
\begin{aligned}
F' &= D\underline{A}^{*'} = D(\underline{A}^* - D^*\Lambda) \\
&= D\underline{A}^* - DD^*\Lambda \\
&= (F + D\circ\underline{A}) - \Box\Lambda \\
&= F - \Box\Lambda
\end{aligned}
\qquad \text{4-6-4}
$$

In order for this to be invariant we need $\Box\Lambda = 0$ i.e. Λ satisfies the wave equation:

$\left(\partial t^2 - \partial x^2 - \partial y^2 - \partial z^2\right)\Lambda = 0$ which can be solved using known methods. So the 4-force is indeed not changed under the gauge transformation.

4-7 Lorentz Gauge

In order for the 4-potential $\underline{A}$ to satisfy Lorentz's condition we require $D \circ \underline{A} = 0$ so, given some 4-potential $\underline{A}$, we have $\Box\Lambda - D \circ \underline{A} = 0$ which is Poisson Equation; which we can solve. We have then a Lorentz gauge. So we have $DF^* = \underline{J}$ and then $F = D\underline{A}^*$ which are now invariant under "Lorentz" gauge transformations satisfying $\Box\Lambda = 0$.

4-8 Relativistic Equation of Motion

Let us now consider the motion of a charged particle with mass m and charge q, being influenced by an EM field. Let it have rapidity β; in a proper time interval $d\tau$, $dt = \left(\cosh\beta\right)dt = \gamma d\tau$. The relativistic equation of motion of the particle is the Lorentz equation of motion: $\frac{d}{dt}m\gamma\vec{v} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$ 4-8-1

or in proper time: $\frac{d}{d\tau}m\gamma\vec{v} = q\gamma\left(\vec{E} + \vec{v} \times \vec{B}\right)$ noting $\gamma = \frac{dt}{d\tau}$ 4-8-2

Let us now convert this to a quaternion form by taking the dot product of both sides with $\gamma\vec{v}$:

$$\gamma\vec{v} \circ \frac{d}{d\tau} m\gamma\vec{v} = q\gamma^2\left(\vec{v} \circ \vec{E} + \vec{v} \circ \left(\vec{v} \times \vec{B}\right)\right)$$

$$m\left(\gamma\vec{v} \circ \frac{d}{d\tau}\gamma\vec{v}\right) = q\gamma^2\vec{v} \circ \vec{E} + 0$$

$$\begin{aligned}
q\gamma^2\vec{v} \circ \vec{E} &= m\frac{1}{2}\frac{d}{d\tau}\left(\gamma^2(\vec{v} \circ \vec{v})\right) = m\frac{1}{2}\frac{d}{d\tau}\gamma^2 v^2 \\
&= m\frac{1}{2}\frac{d}{d\tau}\left(\sinh^2\beta\right) = m\frac{1}{2}\frac{d}{d\tau}\left(\cosh^2\beta - 1\right) \qquad \text{4-8-3} \\
&= m\frac{1}{2}\frac{d}{d\tau}\left(\gamma^2 - 1\right) = m\frac{1}{2}\frac{d}{d\tau}\gamma^2 \\
&= \gamma\frac{d}{d\tau}\gamma m \\
q\gamma\vec{v} \circ \vec{E} &= \frac{d}{d\tau}m\gamma
\end{aligned}$$

If we look at the particle momentum (i.e. Newton's second law)

$$\begin{aligned}
\underline{p} &= m\underline{u} \\
\frac{d\underline{p}}{d\tau} &= \frac{d}{d\tau}m\underline{u} = \frac{d}{d\tau}m\gamma\left(1 + i\vec{v}\right) \qquad \text{4-8-4} \\
&= \frac{d}{d\tau}m\gamma + i\frac{d}{d\tau}m\gamma\vec{v}
\end{aligned}$$

Now substitute the results of equations 4-8-1 and 4-8-3 into equation 4-8-4:

$$\begin{aligned}
\frac{d\underline{p}}{d\tau} &= \frac{d}{d\tau}m\gamma + i\frac{d}{d\tau}m\gamma\vec{v} \\
&= q\gamma\left(\vec{v}\circ\vec{E}\right) + iq\gamma\left(\vec{E} + \vec{v}\times\vec{B}\right) \\
&= q\gamma\left[-\frac{1}{2}\left(\vec{v}\vec{E} + \vec{E}\vec{v}\right) + i\left(\vec{E} + \frac{1}{2}\left(\vec{v}\vec{B} - \vec{B}\vec{v}\right)\right)\right] \\
&= \frac{q\gamma}{2}\left[-\vec{v}\vec{E} - \vec{E}\vec{v} + i2\vec{E} + i\vec{v}\vec{B} - \vec{B}i\vec{v}\right] \\
&= \frac{q\gamma}{2}\left[\left\{2\vec{E}\right\} + \left(i\vec{v}\right)\left(i\vec{E}\right) + \left(i\vec{E}\right)\left(i\vec{v}\right) + i\vec{v}\vec{B} - \vec{B}i\vec{v}\right] \\
&= \frac{q\gamma}{2}\left[\left\{\left(-\vec{B} + i\vec{E}\right) + \left(\vec{B} + i\vec{E}\right)\right\} + i\vec{v}\left(i\vec{E} + \vec{B}\right) + \left(i\vec{E} - \vec{B}\right)i\vec{v}\right] \\
&= \frac{q\gamma}{2}\left[\left(-\vec{B} + i\vec{E}\right)\left(1 + i\vec{v}\right) + \left(1 + i\vec{v}\right)\left(\vec{B} + i\vec{E}\right)\right]
\end{aligned} \qquad \text{4-8-5}$$

Now we know $F = -\vec{B} + i\vec{E}$ and $\underline{u} = \gamma\left(1 + i\vec{v}\right) = \exp\left(i\hat{v}\beta\right)$ so

$$\frac{d\underline{p}}{d\tau} = \frac{q}{2}\left[F\underline{u} + \underline{u}F^{\mathrm{T}}\right] = \frac{q}{2}\left[F\underline{u} - \underline{u}F^{*}\right] \qquad \text{4-8-6}$$

or as $\underline{p} = m\underline{u}$ then $\frac{d\underline{p}}{d\tau} = \frac{q}{2m}\left[F\underline{p} + \underline{p}F^{\mathrm{T}}\right]$ 4-8-7

which is now a quaternion form of the equation of motion in an EM field (via Newton's second law).

4-9 Lorentz Transformation of the Electromagnetic Field and Lorentz's Equation of Motion

Let L represent a Lorentz transformation; e.g. for a passive boost, namely $L = e^{i\hat{n}\frac{\alpha}{2}}$. We already have:

$$\begin{aligned} \underline{x} &\rightarrow L\underline{x}L^{\mathrm{T}} \\ \underline{u} &\rightarrow L\underline{u}L^{\mathrm{T}} \\ D &\rightarrow LDL^{\mathrm{T}} \\ \underline{p} &\rightarrow L\underline{p}L^{\mathrm{T}} \end{aligned} \qquad \text{4-9-1}$$

and we also know that m, q, dτ, and d/ dτ are all invariant under Lorentz transformation.

Now let us apply the L to Lorentz's equation of motion and evaluate each component:

$$\begin{aligned} L\frac{d\underline{p}}{d\tau}L^{\mathrm{T}} &= L\left[\frac{q}{2m}\left(F\underline{p}+\underline{p}F^{\mathrm{T}}\right)\right]L^{\mathrm{T}} \\ \frac{d}{d\tau}L\underline{p}L^{\mathrm{T}} &= \frac{q}{2m}L\left(F\underline{p}+\underline{p}F^{\mathrm{T}}\right)L^{\mathrm{T}} \\ &= \frac{q}{2m}\left[L\left(F\underline{p}\right)L^{\mathrm{T}} + L\left(\underline{p}F^{\mathrm{T}}\right)L^{\mathrm{T}}\right] \\ &= \frac{q}{2m}\left[LFL^{\times}L\underline{p}L^{\mathrm{T}} + L\underline{p}L^{\mathrm{T}}L^{*}F^{\mathrm{T}}L^{\mathrm{T}}\right] \end{aligned} \qquad \text{4-9-2}$$

Then we can conclude that the Lorentz transformation of the Field is $F \rightarrow LFL^{\times}$ or $F^{\mathrm{T}} \rightarrow L^{*}F^{\mathrm{T}}L^{\mathrm{T}}$ as $L\underline{p}L^{\mathrm{T}}$ is indeed the correct transformation for 4-momentum. Similarly we find:

$$\begin{aligned} F^{\times} &\rightarrow LF^{\times}L^{\times} \\ F^{*} &\rightarrow L^{*}F^{*}L^{\mathrm{T}} \end{aligned} \qquad \text{4-9-3}$$

where in the case of the pure passive boost

$$L = \exp\left(-i\hat{n}\frac{\alpha}{2}\right) = L^{\mathrm{T}}$$
$$L^{\times} = \exp\left(i\hat{n}\frac{\alpha}{2}\right) = L^{*} \qquad \text{4-9-4}$$

Maxwell's Equation is now given as $DF^{*} = \underline{J}$. We are using a Lorentz gauge and we have the correct Lorentz transformation for the 4-current as $\underline{J} \rightarrow L\underline{J}L^{\mathrm{T}}$ as well as for the differential field operator as $D \rightarrow LDL^{\mathrm{T}}$. We know how the field should transform so let us simply apply this to Maxwell's equation and see if it is consistent:

$$\begin{aligned} LDF^{*}L^{\mathrm{T}} &= LDL^{\mathrm{T}}L^{*}F^{*}L^{\mathrm{T}} \\ &= LDL^{\mathrm{T}}\left(LFL^{\times}\right)^{*} \end{aligned} \qquad \text{4-9-6}$$

Which is indeed correct. Now look at the conjugate form of Maxwell's equation:

$$\begin{aligned} &D^{*}F = \underline{J}^{*} \\ &\underline{J}^{*} \rightarrow L^{*}\underline{J}^{*}L^{\times} \\ &D^{*}F \rightarrow L^{*}D^{*}L^{\times}LFL^{\times} \\ &\quad \rightarrow \left(LDL^{\mathrm{T}}\right)^{*}LFL^{\times} = \left(LDL^{\mathrm{T}}\left(LFL^{\times}\right)^{*}\right)^{*} \end{aligned} \qquad \text{4-9-7}$$

which is similarly consistent.

4-10 Example of the Lorentz Transformation of the EM Field

Given a constant EM field: $-i\,F = E_{\circ}\,\hat{i} + i\,B_{\circ}\,\hat{j}$ Find the resultant field if it is transformed by $\alpha\; = \ln 3$ in the direction $\hat{n} = \frac{1}{5}\left(-3\,\hat{j} + 4\,\hat{k}\right)$.

Solution: (note that $\cosh\frac{\alpha}{2} = \frac{2}{\sqrt{3}}$, $\sinh\frac{\alpha}{2} = \frac{1}{\sqrt{3}}$)

$$L = e^{-i\frac{\alpha}{2}\hat{n}} = \frac{1}{\sqrt{3}}(2 - i\hat{n}) \qquad \text{4-10-1}$$

$$\begin{aligned}
-iF' &= L(-iF)L^{\times} \\
&= \frac{1}{\sqrt{3}}(2 - i\hat{n})\left(E_{\circ}\hat{i} + iB_{\circ}\hat{j}\right)\frac{1}{\sqrt{3}}(2 + i\hat{n}) \\
&= \frac{1}{3}(2 - i\hat{n})\left(2E_{\circ}\hat{i} + iE_{\circ}\hat{i}\hat{n} + 2iB_{\circ}\hat{j} - B_{\circ}\hat{j}\hat{n}\right) \\
&= \frac{1}{3}\left(4E_{\circ}\hat{i} + 2iE_{\circ}\hat{i}\hat{n} + 4iB_{\circ}\hat{j} - 2B_{\circ}\hat{j}\hat{n} - 2iE_{\circ}\hat{n}\hat{i} + E_{\circ}\hat{n}\hat{i}\hat{n} + 2B_{\circ}\hat{n}\hat{j} + iB_{\circ}\hat{n}\hat{j}\hat{n}\right) \qquad \text{4-10-2} \\
&= \frac{1}{3}\left(4E_{\circ}\hat{i} + 2B_{\circ}\left(\hat{n}\hat{j} - \hat{j}\hat{n}\right) + 4iB_{\circ}\hat{j} + 2iE_{\circ}\left(\hat{i}\hat{n} - \hat{n}\hat{i}\right) + E_{\circ}\hat{n}\hat{i}\hat{n} + iB_{\circ}\hat{n}\hat{j}\hat{n}\right) \\
&= \frac{1}{75}\left(100E_{\circ}\hat{i} + 50B_{\circ}\left(\hat{n}\hat{j} - \hat{j}\hat{n}\right) + 100iB_{\circ}\hat{j} + 50iE_{\circ}\left(\hat{i}\hat{n} - \hat{n}\hat{i}\right) + 25E_{\circ}\hat{n}\hat{i}\hat{n} + 25iB_{\circ}\hat{n}\hat{j}\hat{n}\right)
\end{aligned}$$

note:

$$\begin{aligned}
&\hat{i}\,\hat{n} = -\frac{1}{5}\left(4\hat{j} + 3\hat{k}\right) = -\hat{n}\,\hat{i} \\
&\hat{n}\,\hat{j} = \frac{1}{5}\left(3 - 4\hat{i}\right) \qquad \hat{j}\,\hat{n} = \frac{1}{5}\left(3 + 4\hat{i}\right) \qquad \text{4-10-3} \\
&\hat{n}\,\hat{i}\,\hat{n} = \hat{i} \\
&\hat{n}\,\hat{j}\,\hat{n} = \frac{1}{25}\left(7\hat{j} + 24\hat{k}\right)
\end{aligned}$$

So that

$$
\begin{aligned}
-iF' &= \frac{1}{75}\left(100E_\circ\hat{i} + 50B_\circ(\hat{n}\hat{j} - \hat{j}\hat{n}) + 100iB_\circ\hat{j} + 50iE_\circ(\hat{i}\hat{n} - \hat{n}\hat{i}) + 25E_\circ\hat{n}\hat{i}\hat{n} + 25iB_\circ\hat{n}\hat{j}\hat{n}\right) \\
&= \frac{1}{75}\left(100E_\circ\hat{i} + -80B_\circ\hat{i} + 100iB_\circ\hat{j} - 20iE_\circ(4\hat{j} + 3\hat{k}) + 25E_\circ\hat{i} + iB_\circ(7\hat{j} + 24\hat{k})\right) \\
&= \frac{1}{75}\left((125E_\circ - 80B_\circ)\hat{i} + i\left[-80E_\circ\hat{j} + 107B_\circ\hat{j} + (24B_\circ - 60E_\circ)\hat{k}\right]\right)
\end{aligned}
\qquad \text{4-10-4}
$$

A transformation in an arbitrary direction has been achieved in a straightforward way.

4-11 Energy Density and the Poynting Vector

Let us now find a quaternion form for Poynting's theorem. We assume that the medium is isotropic and linear in magnetic and electrical properties. Using the conservation of energy and momentum we have from the theorem that the time rate of change of the instantaneous average power density is given by minus the divergence of the Poynting vector $P = (\vec{E} \times \vec{B})$. Since our field, F, includes both the electric and magnetic components we can assume that we need only take:

$$
\begin{aligned}
\frac{1}{2}FF^{T} &= \frac{1}{2}(-\vec{B} + i\vec{E})(\vec{B} + i\vec{E}) \\
&= \frac{1}{2}(-\vec{B}\vec{B} - i\vec{B}\vec{E} + i\vec{E}\vec{B} + i^2\vec{E}\vec{E}) \\
&= \frac{1}{2}(B^2 + 2i\vec{E} \times \vec{B} + E^2) \\
&= \frac{1}{2}\left[E^2 + B^2\right] + i\vec{E} \times \vec{B}
\end{aligned}
\qquad \text{4-12-1}
$$

We can recognize the first term as the energy density and the imaginary term as the Poynting Vector ($\vec{S}$).

4-12 The Complex Lorentz Invariant

We have the significance of FF^{T}. Now what about $FF^{\times}$ knowing that $F = -\vec{B} + i\vec{E} = -F^{\times}$? Is this invariant under the Lorentz transformation. We have

$$\begin{aligned} F &\to LFL^{\times} \\ F^{\times} &\to LF^{\times}L^{\times} \end{aligned} \qquad \text{4-12-1}$$

Now $FF^{\times} \to LFL^{\times}LF^{\times}L^{\times} = LFF^{\times}L^{\times}$ so if $FF^{\times}$ is scalar then it is indeed invariant. Recall from chapter 3 that a quaternion Q (real or complex) is scalar iff $Q = Q^{\times}$. So let us check $FF^{\times}$ under this condition:

$$\left(FF^{\times}\right)^{\times} = F^{\times\times}F^{\times} = FF^{\times} \qquad \text{4-12-2}$$

Which is indeed a scalar, and therefore invariant.

$$\therefore FF^{\times} \to LFF^{\times}L^{\times} = FF^{\times} \qquad \text{4-12-3}$$

So $FF^{\times}$ is invariant under the Lorentz transformation. Call it the Lorentz invariant. Now let us evaluate it:

$$\begin{aligned} FF^{\times} &= \left(-\vec{B} + i\vec{E}\right)\left(\vec{B} - i\vec{E}\right) \\ &= B^2 - E^2 + i\vec{B}\vec{E} + \vec{E}\vec{B} \\ &= \left(B^2 - E^2\right) - 2i\vec{B} \circ \vec{E} \end{aligned} \qquad \text{4-12-4}$$

of which both terms are invariant under the Lorentz transformation A more useful form will be

$$\tfrac{1}{2}FF^{\times} = \tfrac{1}{2}\left(B^2 - E^2\right) - i\vec{B} \circ \vec{E} \qquad \text{4-12-5}$$

so the real and imaginary parts here are both invariant under the Lorentz transformation.

5 Electromagnetic Waves in a Vacuum

5-1 Preliminary Results

First define the direction of the Electric field as $\hat{l}$, the direction of the Magnetic field as $\hat{m}$, and the direction of a wave's propagation as $\hat{k}$. These form a right handed system of mutually perpendicular unit vectors (orthogonal triad). Now we need to explore the properties of a projection-type 4-vector, defined as $\underline{k} = 1 + i\hat{k}$, so that we may exploit them later. First note that this is a singular 4-vector:

$$\begin{aligned} \underline{k}\underline{k}^* &= \left(1+i\hat{k}\right)\left(1-i\hat{k}\right) \\ &= 1+\hat{k}\hat{k} \\ &= 1-1 \\ &= 0 \end{aligned} \qquad 5\text{-}1\text{-}1$$

Now a useful and interesting property of this unit 4-vector occurs when it is multiplied by a regular unit vector:

$$\begin{aligned} \hat{k}\underline{k} &= \hat{k}\left(1+i\hat{k}\right) \\ &= \hat{k}+i\hat{k}\hat{k} \\ &= \hat{k}-i \\ &= -i\left(1+i\hat{k}\right) \\ &= -i\underline{k} \end{aligned} \qquad 5\text{-}1\text{-}2$$

and similarly: $i\underline{k} = -\hat{k}\underline{k}$ 5-1-3

Also we may form the unit 4-vector from:

$$
\begin{aligned}
(i\hat{k})\hat{k} &= i\hat{k} - (\hat{k} \circ \hat{k}) \\
&= i\hat{k} + 1 \\
&= \underline{k}
\end{aligned}
\qquad 5\text{-}1\text{-}4
$$

Now we can obtain an unitary property from

$$\tfrac{1}{2}(\underline{k} + \underline{k}^*) = \tfrac{1}{2}(1 + i\hat{k} + 1 - i\hat{k}) = \tfrac{1}{2}(2) = 1 \qquad 5\text{-}1\text{-}5$$

and interestingly the regular unit vector from

$$\tfrac{1}{2}(\underline{k} - \underline{k}^*) = \tfrac{1}{2}(1 + i\hat{k} - 1 + i\hat{k}) = i\hat{k} \qquad 5\text{-}1\text{-}6$$

Now if we take the square of the 4-vector:

$$\underline{k}\underline{k} = \underline{k}^2 = 1 + 2i\hat{k} + \hat{k} \circ \hat{k} = 2 + 2i\hat{k} = 2\underline{k} \qquad 5\text{-}1\text{-}7$$

(Actually $\left(\tfrac{1}{2}\underline{k}\right)^n = \tfrac{1}{2}\underline{k}$, so $\underline{k}^n = 2^{n-1}\underline{k}$)

Lastly let us look at the result of multiplying a quaternion in Euler's form by the 4-vector:

$$
\begin{aligned}
\exp(\hat{k}\theta)\underline{k} &= (\cos\theta + \hat{k}\sin\theta)\underline{k} \\
&= \cos\theta\underline{k} + \hat{k}\underline{k}\sin\theta \\
&= \cos\theta\hat{k} - i\hat{k}\sin\theta \\
&= (\cos\theta - i\sin\theta)\underline{k} \\
&= \exp(-i\theta)\underline{k}
\end{aligned}
\qquad 5\text{-}2\text{-}8
$$

which will be very useful for electromagnetic waves.

Similar results hold equally well for the conjugate of this 4-vector.

5-2 The Wave Equation

Now we need to look for a wave equation for the field and the potential (using the Lorentz gauge). Before constructing a wave look at the wave propagation 4-vector dotted with the x-axis vector and how it is affected by the differential field operator:

$$\begin{aligned}
&\underline{k} \circ \underline{x} = k_0 t - \vec{k} \circ \vec{r} \\
&D(\underline{k} \circ \underline{x}) = (\partial_t - i\nabla)(k_0 t - \vec{k} \circ \vec{r}) \\
&= k_0 - 0 - 0 + i(\hat{i}\partial_x + \hat{j}\partial_{yt} + \hat{k}\partial_z)(k_1 x + k_2 y + k_3 z) \\
&= k_0 + i\vec{k} \\
&= \underline{k}
\end{aligned} \qquad \text{5-2-1}$$

and similarly: $D^*(\underline{k} \circ \underline{x}) = \underline{k}^*$ 5-2-3

How does the differential field operator affect a plane wave?

$$\begin{aligned}
D\exp(i\underline{k} \circ \underline{x}) &= D(\cos(\underline{k} \circ \underline{x}) + i\sin(\underline{k} \circ \underline{x})) \\
&= (-\sin(\underline{k} \circ \underline{x}))(D(\underline{k} \circ \underline{x})) + i(\cos(\underline{k} \circ \underline{x}))(D(\underline{k} \circ \underline{x})) \\
&= i(D(\underline{k} \circ \underline{x}))(\cos(\underline{k} \circ \underline{x}) + i\sin(\underline{k} \circ \underline{x})) \\
&= i\underline{k}\exp(i\underline{k} \circ \underline{x})
\end{aligned} \qquad \text{5-2-4}$$

We find, in fact:

$$\begin{aligned}
D\exp(\pm\hat{k}\underline{k} \circ \underline{x}) &= \mp i\underline{k}\exp(\pm\hat{k}\underline{k} \circ \underline{x}) \\
D^*\exp(\pm\hat{k}\underline{k} \circ \underline{x}) &= \pm i\underline{k}^*\exp(\pm\hat{k}\underline{k} \circ \underline{x})
\end{aligned} \qquad \text{5-2-5}$$

These should provide a solution to the wave equation:

$$\begin{aligned}\Box \exp(i\underline{k}\circ\underline{x}) &= DD^* \exp(i\underline{k}\circ\underline{x}) \\ &= D\left(i\underline{k}^* \exp(i\underline{k}\circ\underline{x})\right) \\ &= \left(D\exp(i\underline{k}\circ\underline{x})\right)(i\underline{k}^*) \\ &= i\underline{k}\exp(i\underline{k}\circ\underline{x})(i\underline{k}^*) \\ &= i\underline{k}i\underline{k}^* \exp(i\underline{k}\circ\underline{x}) \\ &\doteq -\underline{k}\underline{k}^* \exp(i\underline{k}\circ\underline{x}) \\ &= 0\end{aligned} \qquad 5\text{-}2\text{-}6$$

Thus $\exp(\pm i\underline{k}\circ\underline{x})$ represents a scalar plane wave propagating with velocity c = 1 in the direction $\hat{k} = \frac{\vec{k}}{k_0}$. This satisfies the wave equation.

5-3 Plane Wave Solution of the Wave Equation

In a vacuum Maxwell's equation reads: $\underline{J} = 0$. Therefore $DF^* = 0 = D^*F$ where $\exists \underline{A}$ such that $F = D\underline{A}^*$ and $D\circ\underline{A} = 0$.

Thus we have $D^*F = 0 \rightarrow 0 = DD^*F = \Box F$ where each component of F satisfies the wave equation. Also since $F^* = D^*\underline{A}$ then $0 = DF^* = D(D^*\underline{A}) = \Box \underline{A}$ so that $\underline{A}$ also satisfies the wave equation. This will be used as it is easier to solve for $\underline{A}$ and then find the field.

Referring to equation 5-2-6, a plane wave solution to the wave equation will have the form:

$$\underline{A} = \exp(i\underline{k}\circ\underline{x})Q_1 + \exp(-i\underline{k}\circ\underline{x})Q_2 \qquad 5\text{-}3\text{-}1$$

where Q_1, Q_2 are constant complex quaternions and $\hat{k}$ is the direction of propagation.

For $\underline{A}$ to be a 4-vector we must have the requirement that $\underline{A} = \underline{A}^{\mathrm{T}}$ (hermitean)

$$\therefore \underline{A} = \exp(i\underline{k} \circ \underline{x})Q_1 + \exp(-i\underline{k} \circ \underline{x})Q_2 = \underline{A}^{\mathrm{T}} = \exp(-i\underline{k} \circ \underline{x})Q_1^{\mathrm{T}} + \exp(i\underline{k} \circ \underline{x})Q_2^{\mathrm{T}} \qquad 5\text{-}3\text{-}2$$

To hold for all x: $Q_1 = Q_2^{\mathrm{T}}, Q_2 = Q_1^{\mathrm{T}}$ must be true.

But since we have arbitrary complex quaternions we can choose to express them in the form:

$Q_1 = c_1 + i\bar{B}_1 - i(c_2 + i\bar{B}_2)$ where $\bar{B}_1 = B_{11}\hat{k} + B_{12}\hat{l} + B_{13}\hat{m}$ and $\hat{k}, \hat{l}$, and $\hat{m}$ are mutually orthogonal.

We can pair c_1 and B_{11} by using the properties of the unit 4-vector and equations 5-1-5 and 5-1-6:

$$1 = \tfrac{1}{2}(\underline{k} + \underline{k}^*) \qquad\qquad i\hat{k} = \tfrac{1}{2}(\underline{k} - \underline{k}^*)$$

So

$$\begin{aligned}
Q_1 &= [c_1 + i\bar{B}_1] - i[c_2 + i\bar{B}_2] \\
&= \left[\tfrac{1}{2}c_1(\underline{k} + \underline{k}^*) + \tfrac{1}{2}B_{11}(\underline{k} - \underline{k}^*) + i(B_{12}\hat{l} + B_{13}\hat{m})\right] \\
&\quad - i\left[\tfrac{1}{2}c_2(\underline{k} + \underline{k}^*) + \tfrac{1}{2}B_{21}(\underline{k} - \underline{k}^*) + i(B_{22}\hat{l} + B_{23}\hat{m})\right] \\
&= \left[\tfrac{1}{2}\underline{k}(c_1 + B_{11}) + \tfrac{1}{2}\underline{k}^*(c_1 - B_{11}) + i(B_{12}\hat{l} + B_{13}\hat{m})\right] \\
&\quad - i\left[\tfrac{1}{2}\underline{k}(c_2 + B_{21}) + \tfrac{1}{2}\underline{k}^*(c_2 - B_{21}) + i(B_{22}\hat{l} + B_{23}\hat{m})\right] \\
&= \left[a_1\underline{k} + a_2\underline{k}^* + iB_{12}\hat{l} + iB_{13}\hat{m}\right] - i\left[a_3\underline{k} + a_4\underline{k}^* + iB_{22}\hat{l} + iB_{23}\hat{m}\right] \\
&= \underline{A}_1 - i\underline{A}_2
\end{aligned} \qquad 5\text{-}3\text{-}3$$

Then $\underline{A} = \exp(i\underline{k} \circ \underline{x})\tfrac{1}{2}(\underline{A}_1 - i\underline{A}_2) + \exp(-i\underline{k} \circ \underline{x})\tfrac{1}{2}(\underline{A}_1 + i\underline{A}_2)$ 5-3-4

since $\underline{A}_1 = \underline{A}_1^{\mathrm{T}}$ and $\underline{A}_2 = \underline{A}_2^{\mathrm{T}}$:

$$\underline{A} = \tfrac{1}{2}\left(\cos(\underline{k}\circ\underline{x}) + i\sin(\underline{k}\circ\underline{x})\right)\underline{A}_1 - \tfrac{1}{2}\left(i\cos(\underline{k}\circ\underline{x}) - \sin(\underline{k}\circ\underline{x})\right)\underline{A}_2$$
$$+\tfrac{1}{2}\left(\cos(\underline{k}\circ\underline{x}) - i\sin(\underline{k}\circ\underline{x})\right)\underline{A}_1 + \tfrac{1}{2}\left(i\cos(\underline{k}\circ\underline{x}) + \sin(\underline{k}\circ\underline{x})\right)\underline{A}_2 \qquad \text{5-3-5}$$
$$= \cos(\underline{k}\circ\underline{x})\underline{A}_1 + \sin(\underline{k}\circ\underline{x})\underline{A}_2$$

and then also $\underline{A}^* = \cos(\underline{k}\circ\underline{x})\underline{A}_1^* + \sin(\underline{k}\circ\underline{x})\underline{A}_2^*$ 5-3-6

Now we can find the field:

$$F = D\underline{A}^*$$
$$= -\sin(\underline{k}\circ\underline{x})\underline{A}_1^* D(\underline{k}\circ\underline{x}) + \cos(\underline{k}\circ\underline{x})\underline{A}_2^* D(\underline{k}\circ\underline{x}) \qquad \text{5-3-7}$$
$$= \underline{k}\left(-\sin(\underline{k}\circ\underline{x})\underline{A}_1^* + \cos(\underline{k}\circ\underline{x})\underline{A}_2^*\right)$$

also a solution to the wave equation. Now let us try to simplify this field by substituting the form of the $\underline{A}_1$ and $\underline{A}_2$ into the above equation:

$$F = \underline{k}\left(-\sin(\underline{k}\circ\underline{x})\underline{A}_1^* + \cos(\underline{k}\circ\underline{x})\underline{A}_2^*\right)$$
$$= \underline{k}\left(-\sin(\underline{k}\circ\underline{x})\left[a_1\underline{k}^* + a_2\underline{k} + i\left(B_{12}\hat{l} + B_{13}\hat{m}\right)\right]\right.$$
$$\left. + \cos(\underline{k}\circ\underline{x})\left[a_3\underline{k}^* + a_4\underline{k} + i\left(B_{22}\hat{l} + B_{23}\hat{m}\right)\right]\right)$$
$$= -\sin(\underline{k}\circ\underline{x})\left[a_1\underline{k}\underline{k}^* + a_2\underline{k}\underline{k} + i\underline{k}\left(B_{12}\hat{l} + B_{13}\hat{m}\right)\right] \qquad \text{5-3-8}$$
$$+ \cos(\underline{k}\circ\underline{x})\left[a_3\underline{k}\underline{k}^* + a_4\underline{k}\underline{k} + i\underline{k}\left(B_{22}\hat{l} + B_{23}\hat{m}\right)\right]$$
$$= -\sin(\underline{k}\circ\underline{x})\left[a_2\underline{k}\underline{k} + i\underline{k}\left(B_{12}\hat{l} + B_{13}\hat{m}\right)\right] + \cos(\underline{k}\circ\underline{x})\left[a_4\underline{k}\underline{k} + i\underline{k}\left(B_{22}\hat{l} + B_{23}\hat{m}\right)\right]$$

As $\underline{k}\underline{k}^* = k_0(0)$. Now $\underline{k}\underline{k}$ generates a non-vector term so we must set $a_2 = a_4 = 0$ or the vector nature of F will be violated. Thus we are left with

$$
\begin{aligned}
F &= -\sin(\underline{k}\circ\underline{x})\left[i\underline{k}(B_{12}\hat{l}+B_{13}\hat{m})\right]+\cos(\underline{k}\circ\underline{x})\left[i\underline{k}(B_{22}\hat{l}+B_{23}\hat{m})\right] \\
&= i\underline{k}\left(-B_{12}\hat{l}\sin(\underline{k}\circ\underline{x})-B_{13}\hat{m}\sin(\underline{k}\circ\underline{x})+B_{22}\hat{l}\cos(\underline{k}\circ\underline{x})+B_{23}\hat{m}\cos(\underline{k}\circ\underline{x})\right) \\
&= i\underline{k}\left(B_{22}\hat{l}\cos(\underline{k}\circ\underline{x})-B_{13}\hat{m}\sin(\underline{k}\circ\underline{x})+B_{23}\hat{m}\cos(\underline{k}\circ\underline{x})-B_{12}\hat{l}\sin(\underline{k}\circ\underline{x})\right) \\
&= i\underline{k}\left(\left[B_{22}\cos(\underline{k}\circ\underline{x})+B_{13}\hat{k}\sin(\underline{k}\circ\underline{x})\right]\hat{l}+\left[B_{23}\cos(\underline{k}\circ\underline{x})-B_{12}\hat{k}\sin(\underline{k}\circ\underline{x})\right]\hat{m}\right) \\
&= i\underline{k}\left(\left[B_{22}\frac{e^{\hat{k}(\underline{k}\circ\underline{x})}+e^{-\hat{k}(\underline{k}\circ\underline{x})}}{2}+B_{13}\hat{k}\frac{e^{\hat{k}(\underline{k}\circ\underline{x})}-e^{-\hat{k}(\underline{k}\circ\underline{x})}}{2}\right]\hat{l}\right. \\
&\qquad\left.+\left[B_{23}\frac{e^{\hat{k}(\underline{k}\circ\underline{x})}+e^{-\hat{k}(\underline{k}\circ\underline{x})}}{2}-B_{12}\hat{k}\frac{e^{\hat{k}(\underline{k}\circ\underline{x})}-e^{-\hat{k}(\underline{k}\circ\underline{x})}}{2}\right]\hat{m}\right) \\
&= i\underline{k}\left(\left[\frac{B_{22}+B_{13}}{2}e^{\hat{k}(\underline{k}\circ\underline{x})}+\frac{B_{22}-B_{13}}{2}e^{-\hat{k}(\underline{k}\circ\underline{x})}\right]\hat{l}\right. \\
&\qquad\left.+\left[\frac{B_{23}-B_{12}}{2}e^{\hat{k}(\underline{k}\circ\underline{x})}+\frac{B_{23}+B_{12}}{2}e^{-\hat{k}(\underline{k}\circ\underline{x})}\right]\hat{m}\right) \\
&= i\underline{k}\left(\left(\frac{B_{22}+B_{13}}{2}\hat{l}+\frac{B_{23}-B_{12}}{2}\hat{m}\right)e^{\hat{k}(\underline{k}\circ\underline{x})}+\left(\frac{B_{22}-B_{13}}{2}\hat{l}+\frac{B_{23}+B_{12}}{2}\hat{m}\right)e^{-\hat{k}(\underline{k}\circ\underline{x})}\right) \qquad \text{5-3-9} \\
&= i\underline{k}\left((C_1\hat{l}+C_2\hat{m})e^{\hat{k}(\underline{k}\circ\underline{x})}+(C_3\hat{l}+C_4\hat{m})e^{-\hat{k}(\underline{k}\circ\underline{x})}\right) \\
&= i\underline{k}\left(e^{\hat{k}(\underline{k}\circ\underline{x})}\vec{A}_1+e^{-\hat{k}(\underline{k}\circ\underline{x})}\vec{A}_2\right)
\end{aligned}
$$

where $\vec{A}_1, \vec{A}_2 \perp \underline{k}$ thus these vectors are in a plane perpendicular to the direction of propagation.

There are only two cases to consider:

Case 1) $\vec{A}_1 = -\hat{l}$ or $\vec{A}_1 = \hat{m}$ and $\vec{A}_2 = 0$ so, choosing the second case: $\underline{A}^* = -A_0\underline{k}e^{\hat{k}\underline{k}\circ\underline{x}}\hat{m}$

$$F = D\left(-iA_0e^{\hat{k}\underline{k}\circ\underline{x}}\hat{m}\right) = -i\hat{k}\underline{k}A_0e^{\hat{k}\underline{k}\circ\underline{x}}\hat{m} = -\underline{k}A_0e^{\hat{k}\underline{k}\circ\underline{x}}\hat{m} \qquad \text{5-3-10}$$

Case 2) $\vec{A}_2 = \hat{l}$ or $\vec{A}_2 = -\hat{m}$ and $\vec{A}_1 = 0$ so, in the second case: $\underline{A}^* = iA_0\underline{k}e^{-\hat{k}\underline{k}\circ\underline{x}}\hat{m}$

$$F = D\left(iA_0e^{-\hat{k}\underline{k}\circ\underline{x}}\hat{m}\right) = -i\hat{k}\underline{k}A_0e^{-\hat{k}\underline{k}\circ\underline{x}}\hat{m} = -\underline{k}A_0e^{-\hat{k}\underline{k}\circ\underline{x}}\hat{m} \qquad \text{5-3-11}$$

In either case we have

$$\begin{aligned} F &= -A_0 e^{\pm \hat{k} \underline{k} \circ \underline{x}} k_0 \left(1 + i\hat{k}\right) \hat{m} \\ &= A_0 k_0 e^{\pm \hat{k} \underline{k} \circ \underline{x}} \left(-\hat{m} + i\hat{l}\right) \\ &= -B_0 \hat{m} + iE_0 \hat{l} \\ &= -\vec{B} + i\vec{E} \end{aligned} \qquad 5\text{-}3\text{-}12$$

which is indeed correct.

5-4 Circular, Plane and Elliptical Polarization

Now let us try to classify these electromagnetic waves under polarization. We will use standard conventions for this with all of the confusing contradictions (but avoiding Stokes Parameters). This section will also help to summarize the results so far that we found for the general plane wave solution.

Circular Polarization

The simplest case we have two "directions" to consider (here comes confusion)

Table 5-4-1: Circular Polarization

	Positive Helicity or Left Circular Polarization	Negative Helicity or Right Circular Polarization
$\underline{A}$	$iA_0 \exp\left(\hat{k}\underline{k} \circ \underline{x}\right)\hat{m}$	$-iA_0 \exp\left(-\hat{k}\underline{k} \circ \underline{x}\right)\hat{m}$
$F = -\vec{B} + i\vec{E}$	$A_0 k_0 \exp\left(\hat{k}\underline{k} \circ \underline{x}\right)\left(-\hat{m} + i\hat{l}\right)$ $= A_0 k_0 \exp\left(-i\underline{k} \circ \underline{x}\right)\left(-\hat{m} + i\hat{l}\right)$	$A_0 k_0 \exp\left(-\hat{k}\underline{k} \circ \underline{x}\right)\left(-\hat{m} + i\hat{l}\right)$ $= A_0 k_0 \exp\left(i\underline{k} \circ \underline{x}\right)\left(-\hat{m} + i\hat{l}\right)$
$-iF = \vec{E} + i\vec{B}$	$A_0 k_0 \exp\left(\hat{k}\underline{k} \circ \underline{x}\right)\left(\hat{l} + i\hat{m}\right)$ $= A_0 k_0 \exp\left(-i\underline{k} \circ \underline{x}\right)\left(\hat{l} + i\hat{m}\right)$	$A_0 k_0 \exp\left(-\hat{k}\underline{k} \circ \underline{x}\right)\left(\hat{l} + i\hat{m}\right)$ $= A_0 k_0 \exp\left(i\underline{k} \circ \underline{x}\right)\left(\hat{l} + i\hat{m}\right)$

In either case the Magnetic and Electric components can be read off directly.

Linear polarization

In this case both negative and positive components are included in the wave so that

$$
\begin{aligned}
\underline{A} &= \tfrac{1}{2}\left[\underline{A}^{+} + \underline{A}^{-}\right] = \tfrac{1}{2} iA_0\left[\exp\left(\hat{k}\underline{k} \circ \underline{x}\right) - \exp\left(-\hat{k}\underline{k} \circ \underline{x}\right)\right]\hat{m} \\
&= \tfrac{1}{2} iA_0\left[\cos\left(\underline{k} \circ \underline{x}\right) + \hat{k}\sin\left(\underline{k} \circ \underline{x}\right) - \cos\left(\underline{k} \circ \underline{x}\right) + \hat{k}\sin\left(\underline{k} \circ \underline{x}\right)\right]\hat{m} \\
&= iA_0\hat{k}\sin\left(\underline{k} \circ \underline{x}\right)\hat{m} \\
&= A_0\sin\left(\underline{k} \circ \underline{x}\right)\left(-i\hat{l}\right)
\end{aligned}
\qquad \text{5-4-1}
$$

$$F = iA_0 \cos(\underline{k} \circ \underline{x}) D(\underline{k} \circ \underline{x}) \hat{l} = iA_0 \underline{k} \cos(\underline{k} \circ \underline{x})(\hat{l})$$
$$= iA_0 \cos(\underline{k} \circ \underline{x}) k_0 (1 + i\hat{k}) \hat{l} = A_0 k_0 \cos(\underline{k} \circ \underline{x}) i (\hat{l} + i\hat{m}) \qquad \text{5-4-2}$$
$$= A_0 k_0 \cos(\underline{k} \circ \underline{x})(-\hat{m} + i\hat{l})$$

$$-iF = \hat{k}F = i\hat{k}\underline{k} A_0 \cos(\underline{k} \circ \underline{x}) \hat{l} = A_0 \cos(\underline{k} \circ \underline{x}) \underline{k}\hat{l}$$
$$= A_0 \cos(\underline{k} \circ \underline{x})(\hat{l} + i\hat{m}) \qquad \text{5-4-3}$$

Elliptical Polarization

The most complicated polarization occurs when the positive and negative parts of the plane wave no longer have the same amplitude:

$$\underline{A} = i\left[A_1 \exp(\hat{k}\underline{k} \circ \underline{x}) - A_2 \exp(-\hat{k}\underline{k} \circ \underline{x})\right]\hat{m}$$
$$= i\left[(A_1 - A_2)\cos(\underline{k} \circ \underline{x}) + \hat{k}(A_1 + A_2)\sin(\underline{k} \circ \underline{x})\right]\hat{m} \qquad \text{5-4-4}$$

$$F = -i\hat{k}\underline{k}\left[A_1 \exp(\hat{k}\underline{k} \circ \underline{x}) + A_2 \exp(-\hat{k}\underline{k} \circ \underline{x})\right]\hat{m}$$
$$= -i\underline{k}\left[A_1 \exp(\hat{k}\underline{k} \circ \underline{x}) + A_2 \exp(-\hat{k}\underline{k} \circ \underline{x})\right]\hat{k}\hat{m}$$
$$= i\left[A_1 \exp(\hat{k}\underline{k} \circ \underline{x}) + A_2 \exp(-\hat{k}\underline{k} \circ \underline{x})\right]\underline{k}\hat{l} \qquad \text{5-4-5}$$
$$= ik_0\left[A_1 \exp(\hat{k}\underline{k} \circ \underline{x}) + A_2 \exp(-\hat{k}\underline{k} \circ \underline{x})\right](\hat{l} + i\hat{m})$$
$$= k_0\left[(A_1 + A_2)\sin(\underline{k} \circ \underline{x}) + (A_1 - A_2)\hat{k}\sin(\underline{k} \circ \underline{x})\right](-\hat{m} + i\hat{l})$$

$$-iF = k_0\left[A_1 \exp(\hat{k}\underline{k} \circ \underline{x}) + A_2 \exp(-\hat{k}\underline{k} \circ \underline{x})\right](\hat{l} + i\hat{m})$$
$$= k_0\left[(A_1 + A_2)\cos(\underline{k} \circ \underline{x}) + (A_1 - A_2)\hat{k}\sin(\underline{k} \circ \underline{x})\right](\hat{l} + i\hat{m}) \qquad \text{5-4-6}$$

Here the magnetic and electric fields are rotating in the $\hat{l} - \hat{m}$ plane, with angular frequency $\omega = k_0$ and an average amplitude between $A_1 + A_2$ and $A_1 - A_2$. Thus the field vectors trace ellipses with semi-axes $A_1 + A_2$ and $A_1 - A_2$.

This is actually the most general case as there are six possibilities for the amplitude representing various polarizations:

1) $A_1 = 0 \; or \; A_2 = 0$ Negative or Positive Helicity

2) $A_1 = A_2 \; or \; A_1 = -A_2$ Linear Polarization

3) $A_1 > A_2 \; or \; A_1 < A_2$ Positive or Negative Elliptical Rotation (right handed system)

5-5 Energy Density and the Poynting Vector

As in chapter 4 we can find a form for Poynting theorem. Again take $\frac{1}{2}FF^{\mathrm{T}}$.

For a linear polarized wave we find

$$\begin{aligned}
\frac{1}{2}FF^{\mathrm{T}} &= \frac{1}{2}\left(A_0 k_0 \cos(\underline{k} \circ \underline{x})(-\hat{m} + i\hat{l})\right)\left(A_0 k_0 \cos(\underline{k} \circ \underline{x})(\hat{m} + i\hat{l})\right) \\
&= \frac{(A_0 k_0)^2}{2}\cos^2(\underline{k} \circ \underline{x})\left((m^2 + l^2 - \hat{m}i\hat{l} + i\hat{l}\hat{m})\right) \\
&= \frac{(A_0 k_0)^2}{2}\cos^2(\underline{k} \circ \underline{x})\left((2) + 2i\hat{k}\right) \\
&= (A_0 k_0)^2 \cos^2(\underline{k} \circ \underline{x})\left(1 + i\hat{k}\right)
\end{aligned} \qquad \text{5-5-1}$$

and the direction that the energy is directed is in the direction of propagation (Poynting's vector).

For a circular polarization (positive helicity) we find

$$\begin{aligned}
\frac{1}{2}FF^{\mathrm{T}} &= \frac{1}{2}\left(A_0k_0 \exp(\underline{k}\circ\underline{x})\left(-\hat{m}+i\hat{l}\right)\right)\left(A_0k_0 \exp(\underline{k}\circ\underline{x})\left(\hat{m}+i\hat{l}\right)\right) \\
&= \frac{\left(A_0k_0\right)^2}{2}\exp 2(\underline{k}\circ\underline{x})\left(\left(m^2+l^2-\hat{m}i\hat{l}+i\hat{l}\hat{m}\right)\right) \\
&= \frac{\left(A_0k_0\right)^2}{2}\exp 2(\underline{k}\circ\underline{x})\left(2+2i\hat{k}\right) \\
&= \left(A_0k_0\right)^2 \exp 2(\underline{k}\circ\underline{x})\left(1+i\hat{k}\right)
\end{aligned} \qquad \text{5-5-2}$$

Again the term that we interpret as Poynting's vector is in the direction of propagation. We expect that the energy should be directed in the direction that the wave is propagating so that the wave is indeed transmitting power.

6 Electromagnetism: Macroscopic theory

6-1 Maxwell's Equations

Recall in chapter 4 the microscopic form of Maxwell's equations in Heaviside-Lorentz units, with c = 1:

$$\begin{aligned}
&\nabla \circ \vec{E} = \rho \\
&\nabla \circ \vec{B} = 0 \\
&\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \\
&\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0
\end{aligned} \qquad \text{6-1-1}$$

or if $c \neq 1$:

$$\begin{aligned}
&\nabla \circ \vec{E} = \rho \\
&\nabla \circ \vec{B} = 0 \\
&\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{1}{c}\vec{J} \\
&\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0
\end{aligned} \qquad \text{6-1-2}$$

Which with the field defined as

$$F = -\vec{B} + i\vec{E} = i\left(\vec{E} + i\vec{B}\right) \qquad \text{6-1-3}$$

and c =1 lead to a quaternion form for Maxwell's equations as

$$DF^{*} = \underline{J} \Leftrightarrow D^{*}F = \underline{J}^{*} \qquad \text{6-1-4}$$

Now we need to find an equivalent form for the macroscopic theory, with $c \neq 1$.

6-2 Modified Maxwell's Equations

Normally we make notation simpler by defining macroscopic field variables as (in Heaviside Lorentz units):

$$\vec{D} = \in \vec{E} + \vec{P}$$
$$\vec{H} = \tfrac{1}{\mu}\vec{B} - \vec{M} \qquad 6\text{-}2\text{-}1$$

where the new components are the polarization and magnetization. A linear, isotropic, homogeneous medium is assumed. These lead to the macroscopic form of Maxwell's equations, in Heaviside-Lorentz units, in the form:

$$\nabla \circ \vec{D} = \rho$$
$$\nabla \circ \vec{B} = 0$$
$$\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{1}{c}\vec{J} \qquad 6\text{-}2\text{-}2$$
$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

In SI (rationalized MKS) units, Maxwell's equation read:

$$\nabla \circ \vec{D} = \rho$$
$$\nabla \circ \vec{B} = 0$$
$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \qquad 6\text{-}2\text{-}3$$
$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

However these forms are very dependent on the system of units used and we will need to change our definition of the quaternion field, F. Thus a change of the above forms is required in such a way that we can salvage the previous notation and results form the microscopic theory developed in chapter 4. So we modify these equations making some definitions:

From now on, let the speed of light in a vacuum be c_0.

Let: $\lambda = \dfrac{1}{\sqrt{\mu \in}}$ and $\vec{G} = \lambda \vec{B}$ 6-2-4

In Heaviside-Lorentz units, let $c = \lambda c_0$

In SI units, let $c = \lambda$

Under these definitions, and assuming from now on that $\in$ and μ are constant, the equations of 6-2-2 and 6-2-3 are changed in the following manner, for Heaviside-Lorentz units:

$$\nabla \times \vec{E} = -\frac{1}{c_0}\frac{\partial \vec{B}}{\partial t} \longrightarrow \nabla \times \vec{E} = -\frac{1}{\lambda c_0}\frac{\partial \vec{G}}{\partial t} = -\frac{1}{c}\frac{\partial \vec{G}}{\partial t}$$

$$\begin{aligned}\frac{1}{c_0}\vec{J} &= \nabla \times \vec{H} - \frac{1}{c_0}\frac{\partial \vec{D}}{\partial t} \\ &= \frac{1}{\mu}\nabla \times \vec{B} - \frac{\in}{c_0}\frac{\partial \vec{E}}{\partial t} \\ &= \frac{1}{\lambda\mu}\left[\nabla \times \lambda\vec{B} - \frac{1}{\lambda c_0}\frac{\partial \vec{E}}{\partial t}\right] \longrightarrow \nabla \times \vec{G} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{1}{\in c}\vec{J}\end{aligned}$$ 6-2-5

So that we now have for Maxwell's equations in matter these modified results, in both the Heaviside Lorentz System and in the SI System:

$$\begin{aligned}&\nabla \circ \vec{E} = \frac{1}{\in}\rho \\ &\nabla \circ \vec{G} = 0 \\ &\nabla \times \vec{G} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{1}{\in c}\vec{J} \\ &\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{G}}{\partial t} = 0\end{aligned}$$ 6-2-6

Finally, define $\tau = ct$ (where this τ is not to be confused with the proper time in Chapter 3).

These redefinitions let us preserve the quaternion notation developed in chapter 4. We also have the added bonus that the above forms are preserved in the rationalized MKS system with the appropriate constant definitions as above.

6-3 Quaternion Form

Now we can follow the same form developed in chapter 4 and define; in either the Heaviside-Lorentz or the Rationalized MKS system of units:

$$\mathbf{F} = -\vec{G} + i\vec{E} = -\lambda\vec{B} + i\vec{E}$$
$$\mathbf{J} = \frac{1}{\in}\left(\rho + \frac{1}{c}i\vec{J}\right) \qquad \text{6-3-1}$$
$$\mathbf{D} = \frac{1}{c}\partial_t - i\nabla = \partial_\tau - i\nabla$$

Then Maxwell's equation in matter read (scriptic lettering)

$$\mathbf{D}\mathbf{F}^* = \mathbf{J} \Leftrightarrow \mathbf{D}^*\mathbf{F} = \mathbf{J}^* \qquad \text{6-3-2}$$

The potential, following the same development in chapter 4, follows from:

$$\nabla \circ \vec{G} = 0 \Rightarrow \exists \vec{\mathbf{A}} \ni \nabla \times \vec{\mathbf{A}} = \vec{G}$$
$$0 = \nabla \times \vec{E} + \frac{\partial \vec{G}}{\partial \tau} = \nabla \times \left(\vec{E} + \frac{\partial \vec{\mathbf{A}}}{\partial \tau}\right) \Rightarrow \exists \Phi \ni \vec{E} = -\nabla\Phi - \frac{\partial \vec{\mathbf{A}}}{\partial \tau} \qquad \text{6-3-3}$$

Now define $\underline{\mathbf{A}} = \Phi + i\vec{\mathbf{A}}$.

Then:

$$\begin{aligned}\mathbf{D}\underline{\mathbb{A}}^{*} &= \left(\partial_{\tau} - i\nabla\right)\left(\Phi - i\vec{\mathbb{A}}\right) \\ &= -\nabla\times\vec{\mathbb{A}} + i\left(-\nabla\Phi - \frac{\partial\vec{\mathbb{A}}}{\partial\tau}\right) + \left(\partial_{\tau}\Phi + \nabla\circ\vec{\mathbb{A}}\right) \\ &= -\vec{G} + i\vec{E} + \mathbf{D}\circ\underline{\mathbb{A}} \\ &= \mathbb{F} + \mathbf{D}\circ\underline{\mathbb{A}}\end{aligned} \qquad \text{6-3-4}$$

But by the same argument as before, we can impose the Lorentz condition $\mathbf{D}\circ\underline{\mathbb{A}} = 0$. So we have indeed the same form as before, namely:

$$\mathbf{D}\underline{\mathbb{A}}^{*} = \mathbb{F} \qquad \text{6-3-5}$$

Now we have a form of Maxwell's equation that will work for all linear isotropic homogeneous media with constant electromagnetic properties, and with time varying fields.

7 Plane E-M Waves in a Linear, Isotropic, Homogeneous Medium

7-1 The Wave Equation

From chapter 5 we need to modify this wave equation

$$\Box \exp(i\underline{k} \circ \underline{x}) = DD^* \exp(i\underline{k} \circ \underline{x}) = 0 \qquad 7\text{-}1\text{-}1$$

to be valid in matter.

In order to accomplish this we redefine the unit 4-vector of propagation to now be:

$$\underline{k} = \frac{\omega}{c} + i\vec{k} = k_0\left(1 + i\hat{k}\right) \qquad 7\text{-}1\text{-}2$$

where $k_0 = |\vec{k}| = \frac{\omega}{c}$

Then all of the unit 4-vector properties remain the same and it is still singular. However the dot product of this with the x-axis vector changes slightly (as expected):

$$\begin{aligned} \underline{k} \circ \underline{x} &= \left(\frac{\omega}{c} + i\vec{k}\right) \circ (ct + i\vec{x}) \\ &= \frac{\omega}{c}ct - \vec{k} \circ \vec{x} \\ &= k_0\tau - \vec{k} \circ \vec{x} \end{aligned} \qquad 7\text{-}1\text{-}3$$

there is now an angular frequency involved.

Does the effect of the differential field operator change? Consider

$$\mathbf{D} \exp\left(\pm \hat{k}\underline{k} \circ \underline{x}\right) = \pm \hat{k}\underline{k} \exp\left(\pm \hat{k}\underline{k} \circ \underline{x}\right) \qquad 7\text{-}1\text{-}4$$

which has not changed in form. Now since all of the results are in the same form we can write down the generalized wave equation directly:

$$\Box \exp\left(\hat{k}\underline{k} \circ \underline{x}\right) = \mathbb{D}\mathbb{D}^{*} \exp\left(\hat{k}\underline{k} \circ \underline{x}\right) = 0 \qquad \text{7-1-5}$$

which now represents a wave propagating in matter.

7-2 Circular, Plane, and Elliptical Polarized Plane Waves

Now classify these waves under polarization. Again using standard conventions.

Circular Polarization

Table 7-2-1: Circular Polarization

	Positive Helicity or Left Circular Polarization	Negative Helicity or Right Circular Polarization
$\underline{A}'$	$iA_0 \exp\left(\hat{k}\underline{k} \circ \underline{x}\right)\hat{m}$	$-iA_0 \exp\left(-\hat{k}\underline{k} \circ \underline{x}\right)\hat{m}$
$\mathbb{F} = -\vec{G} + i\vec{E}$	$A_0 k_0 \exp\left(\hat{k}\underline{k} \circ \underline{x}\right)\left(-\hat{m} + i\hat{l}\right)$ $= A_0 k_0 \exp\left(-i\underline{k} \circ \underline{x}\right)\left(-\hat{m} + i\hat{l}\right)$	$A_0 k_0 \exp\left(-\hat{k}\underline{k} \circ \underline{x}\right)\left(-\hat{m} + i\hat{l}\right)$ $= A_0 k_0 \exp\left(i\underline{k} \circ \underline{x}\right)\left(-\hat{m} + i\hat{l}\right)$
$-i\mathbb{F} = \vec{E} + i\vec{G}$	$A_0 k_0 \exp\left(\hat{k}\underline{k} \circ \underline{x}\right)\left(\hat{l} + i\hat{m}\right)$ $= A_0 k_0 \exp\left(-i\underline{k} \circ \underline{x}\right)\left(\hat{l} + i\hat{m}\right)$	$A_0 k_0 \exp\left(-\hat{k}\underline{k} \circ \underline{x}\right)\left(\hat{l} + i\hat{m}\right)$ $= A_0 k_0 \exp\left(i\underline{k} \circ \underline{x}\right)\left(\hat{l} + i\hat{m}\right)$

No changes here from chapter 5.

Linear polarization

In this case both negative and positive components are included in the wave so that

$$\begin{aligned}\underline{A}' &= \tfrac{1}{2}\left[\underline{A}'^{+} + \underline{A}'^{-}\right] = \tfrac{1}{2} i A_0\left[\exp\left(\hat{k}\underline{k}\circ\underline{x}\right) - \exp\left(-\hat{k}\underline{k}\circ\underline{x}\right)\right]\hat{m} \\ &= \tfrac{1}{2} i A_0\left[\cos\left(\underline{k}\circ\underline{x}\right) + \hat{k}\sin\left(\underline{k}\circ\underline{x}\right) - \cos\left(\underline{k}\circ\underline{x}\right) + \hat{k}\sin\left(\underline{k}\circ\underline{x}\right)\right]\hat{m} \\ &= i A_0 \hat{k}\sin\left(\underline{k}\circ\underline{x}\right)\hat{m} \\ &= A_0 \sin\left(\underline{k}\circ\underline{x}\right)\left(-i\hat{l}\right)\end{aligned} \qquad \text{7-2-1}$$

$$\begin{aligned}\mathbf{F} = \mathbf{D}\underline{A}' &= i A_0 \cos\left(\underline{k}\circ\underline{x}\right)\mathbf{D}\left(\underline{k}\circ\underline{x}\right)\hat{l} \\ &= i A_0 \underline{k}\cos\left(\underline{k}\circ\underline{x}\right)\left(\hat{l}\right) \\ &= A_0 k_0 \cos\left(\underline{k}\circ\underline{x}\right) i\left(\hat{l} + i\hat{m}\right) \\ &= A_0 k_0 \cos\left(\underline{k}\circ\underline{x}\right)\left(-\hat{m} + i\hat{l}\right)\end{aligned} \qquad \text{7-2-2}$$

$$\begin{aligned}-i\mathbf{F} = \hat{k}\mathbf{F} &= \vec{E} + i\vec{G} \\ &= i\hat{k}\underline{k}A_0 \cos\left(\underline{k}\circ\underline{x}\right)\hat{l} \\ &= A_0 \cos\left(\underline{k}\circ\underline{x}\right)\underline{k}\hat{l} \\ &= A_0 k_0 \cos\left(\underline{k}\circ\underline{x}\right)\left(\hat{l} + i\hat{m}\right)\end{aligned} \qquad \text{7-2-3}$$

Elliptical Polarization

$$\begin{aligned}\underline{A}' &= i\left[A_1 \exp\left(\hat{k}\underline{k}\circ\underline{x}\right) - A_2 \exp\left(-\hat{k}\underline{k}\circ\underline{x}\right)\right]\hat{m} \\ &= i\left[\left(A_1 - A_2\right)\cos\left(\underline{k}\circ\underline{x}\right) + \hat{k}\left(A_1 + A_2\right)\sin\left(\underline{k}\circ\underline{x}\right)\right]\hat{m}\end{aligned} \qquad \text{7-2-4}$$

$$\begin{aligned}\mathbf{F} &= -i\hat{k}\underline{k}\left[A_1 \exp\left(\hat{k}\underline{k}\circ\underline{x}\right) + A_2 \exp\left(-\hat{k}\underline{k}\circ\underline{x}\right)\right]\hat{m} \\ &= -i\underline{k}\left[A_1 \exp\left(\hat{k}\underline{k}\circ\underline{x}\right) + A_2 \exp\left(-\hat{k}\underline{k}\circ\underline{x}\right)\right]\hat{k}\hat{m} \\ &= i\left[A_1 \exp\left(\hat{k}\underline{k}\circ\underline{x}\right) + A_2 \exp\left(-\hat{k}\underline{k}\circ\underline{x}\right)\right]\underline{k}\hat{l} \\ &= i k_0\left[A_1 \exp\left(\hat{k}\underline{k}\circ\underline{x}\right) + A_2 \exp\left(-\hat{k}\underline{k}\circ\underline{x}\right)\right]\left(\hat{l} + i\hat{m}\right) \\ &= k_0\left[\left(A_1 + A_2\right)\sin\left(\underline{k}\circ\underline{x}\right) + \left(A_1 - A_2\right)\hat{k}\sin\left(\underline{k}\circ\underline{x}\right)\right]\left(-\hat{m} + i\hat{l}\right)\end{aligned} \qquad \text{7-2-5}$$

$$
\begin{aligned}
-i\mathbb{F} &= k_0\left[A_1 \exp\left(\hat{k}\underline{k} \circ \underline{x}\right) + A_2 \exp\left(-\hat{k}\underline{k} \circ \underline{x}\right)\right]\left(\hat{l} + i\hat{m}\right) \\
&= k_0\left[\left(A_1 + A_2\right)\cos\left(\underline{k} \circ \underline{x}\right) + \left(A_1 - A_2\right)\hat{k}\sin\left(\underline{k} \circ \underline{x}\right)\right]\left(\hat{l} + i\hat{m}\right)
\end{aligned}
\qquad 7\text{-}2\text{-}6
$$

Here the magnetic and electric fields are rotating in the $\hat{l} - \hat{m}$ plane, with angular frequency, $\frac{\omega}{c} = k_0$ and an average amplitude between $A_1 + A_2$ and $A_1 - A_2$. Now we see the true power of the quaternion notation as we have a consistent equation for a wave in a medium or in a vacuum.

8 Transmission Lines

8-1 Solution to the Wave Transmission Problem with a Vacuum Annulus

It is more convenient to use the form of Maxwell's equation for the complex field operator namely $D^*(iF) = iD^*F = 0$. For this reason now define $\vec{g} = -iF = \vec{E} + i\vec{B}$ so that $\vec{g}$ is now a complex vector. Also it is convenient to define two partial differential operators as

$$\nabla_2 = \hat{i}\partial_x + \hat{j}\partial_y \text{ such that } \nabla = \hat{k}\partial_z + \nabla_2 \qquad \text{8-1-1}$$

$$\text{and} \qquad \Delta_2 = \partial_{xx} + \partial_{yy} \qquad \text{8-1-2}$$

We have Maxwell's equation in a vacuum as $D^*F = 0 \Leftrightarrow D^*\vec{g} = 0$. Now let us make an assumption of the form of the solution as

$$\vec{g} = \cos(\omega t - \beta z)\vec{g}_1 + \sin(\omega t - \beta z)\vec{g}_2 \qquad \text{8-1-3}$$

such that $\vec{g}_i \equiv \vec{g}_i(x, y)$, which represents an electromagnetic wave propagating in the $\hat{k}$ direction. Now substitute this assumed form of the solution into Maxwell's equation:

$$\begin{aligned} D^*\vec{g} &= (\partial_t + i\nabla)(\cos(\omega t - \beta z)\vec{g}_1 + \sin(\omega t - \beta z)\vec{g}_2) \\ &= -\sin(\omega t - \beta z)\left[(\omega - i\beta\hat{k})\vec{g}_1 - i\nabla_2\vec{g}_2\right] \\ &\quad + \cos(\omega t - \beta z)\left[(\omega - i\beta\hat{k})\vec{g}_2 + i\nabla_2\vec{g}_1\right] \end{aligned} \qquad \text{8-1-4}$$

for a solution to exist: $\left[(\omega - i\beta\hat{k})\vec{g}_1 - i\nabla_2\vec{g}_2\right] = 0$ 8-1-5

and $\left[(\omega - i\beta\hat{k})\vec{g}_2 + i\nabla_2\vec{g}_1\right] = 0$ must be true. 8-1-6

One possibility is $\nabla_2 \vec{g}_1 = 0$ and $\nabla_2 \vec{g}_2 = 0$ which implies:

1) $\left(\nabla_2 \times \vec{g}_i\right) = 0 \longrightarrow \vec{g}_i = -\nabla_2 \phi_i$ 8-1-7

2) $\left(\nabla_2 \circ \vec{g}_i\right) = 0 \longrightarrow \Delta_2 \phi_i = 0$ 8-1-8

3) $\left(\omega t - i\beta \hat{k}\right)\vec{g}_i = 0$ 8-1-9

where in equation 8-1-9 both factors are complex quaternions. Refer to chapter 3. This implies that both quaternions are either singular or equal to zero. Thus for a non-trivial solution both must be singular. $\vec{g}_i$ is fine as it is arbitrary by definition and thus can be defined to be singular. $\left(\omega t - i\beta \hat{k}\right)$ must be made singular. Thus: $\left(\omega t - i\beta \hat{k}\right) = k_0\left(1 - i\hat{k}\right)$ to be singular so $\omega = \beta = k_0$.

These three requirements will let us define a form for $\vec{g}_i = \left(1 + i\hat{k}\right)Q_i$ where Q_i is a general quaternion, say: $Q_i = \left[(a + ib) + \left(c_1\hat{k} + ic_2\hat{k}\right) + \left(\vec{A} + i\vec{B}\right)\right]$ 8-1-10

where $\vec{A}, \vec{B} \perp \hat{k}$

Now $(a + ib)$ would generate scalar terms which do not make sense for a vectorial solution. Also $\left(c_1\hat{k} + ic_2\hat{k}\right)$ generates scalar terms since $\hat{k}^2 = -1$. Thus we are left with $Q_i = \left(\vec{A} + i\vec{B}\right)$ which produces vectorial terms as required.

So

$$\begin{aligned}(1+i\hat{k})(\vec{A}+i\vec{B}) &= (\vec{A}-\hat{k}\times\vec{B})+i(\hat{k}\times\vec{A}+\vec{B}) \\ &= (\vec{A}-\hat{k}\vec{B})+i\hat{k}(\vec{A}-\hat{k}\vec{B}) \\ &= (1+i\hat{k})(\vec{A}-\hat{k}\vec{B}) \\ &-(1+i\hat{k})\vec{E}_0\end{aligned} \quad \text{8-1-11}$$

Thus $\vec{g}_i = \vec{E}_{0i}$ a complex vector.

We have solution for Maxwell's equation in a vacuum as:

$$\vec{g} = \left((\cos k_0(t-z))\vec{E}_{01} + (\sin k_0(t-z))\vec{E}_{02}\right)(1+i\hat{k}) \quad \text{8-1-12}$$

We can choose the origin such that the term $(\sin k_0(t-z))\vec{E}_{02}(1+i\hat{k})$ is absent. Then

$$\begin{aligned}\vec{g} &= (\cos k_0(t-z))(1+i\hat{k})\vec{E}_0 \\ &= (\cos k_0(t-z))\vec{E}_0(1-i\hat{k})\end{aligned} \quad \text{8-1-13}$$

where $\vec{E}_0 \perp \hat{k}$, $\nabla_2 \times \vec{E}_0 = 0$, and $\nabla_2 \circ \vec{E}_0 = 0$.

Then $\vec{E}_0 = -\nabla_2\phi$ and $\Delta_2\phi = 0$ where ϕ is constant on the conducting boundaries defining the transmission line.

So $\vec{E} = (\cos k_0(t-z))\vec{E}_0$ and $\vec{B} = \hat{k}\vec{E}$

where $\vec{B} \perp \vec{E}$; $|\vec{B}| = |\vec{E}|$ which we expect for a wave solution.

We should also be able to derive this solution from the 4-potetnial. Recall

$$F = D\underline{A}^* \Leftrightarrow \vec{g} = -iD\underline{A}^* \quad \text{8-1-14}$$

where $\underline{A} = (\cos k_0(t-z))\phi(1+i\hat{k})$

Then

$$
\begin{aligned}
D\underline{A}^{*} &= (\partial_t - i\nabla)\left[(\cos k_0(t-z))\phi(1-i\hat{k})\right] \\
&= \left[\partial_t - i\hat{k}\partial_z - i\nabla_2\right](\cos k_0(t-z))\phi(1-i\hat{k}) \\
&= -k_0(\sin k_0(t-z))\phi\left[(1+i\hat{k})(1-i\hat{k})\right] - i(\cos k_0(t-z))(\nabla_2\phi)(1-i\hat{k}) \\
&= 0 - i(\cos k_0(t-z))(-\vec{E}_0)(1-i\hat{k}) \\
&= i(\cos k_0(t-z))(\vec{E}_0)(1-i\hat{k}) \qquad \text{8-1-15} \\
&= i(\cos k_0(t-z))(1+i\hat{k})\vec{E}_0 \\
&= i(\cos k_0(t-z))(\vec{E}_0 + i\hat{k}\vec{E}_0) \\
&= (\cos k_0(t-z))(i\vec{E}_0 - \vec{B}_0) \\
&= F
\end{aligned}
$$

As required!

8-2 Conformal Mapping as a Method of Solving Boundary Value Problems

Since $\vec{E}$ is analytic and $\nabla\vec{E} \neq 0$ we should be able to apply complex conformal mapping as a method of solving the boundary value problem for transmission lines. Refer to figure 8-2-1 representing a cross-section of a very long transmission line and it's mapping to the W-plane. Note the branch cut on the negative x-axis.

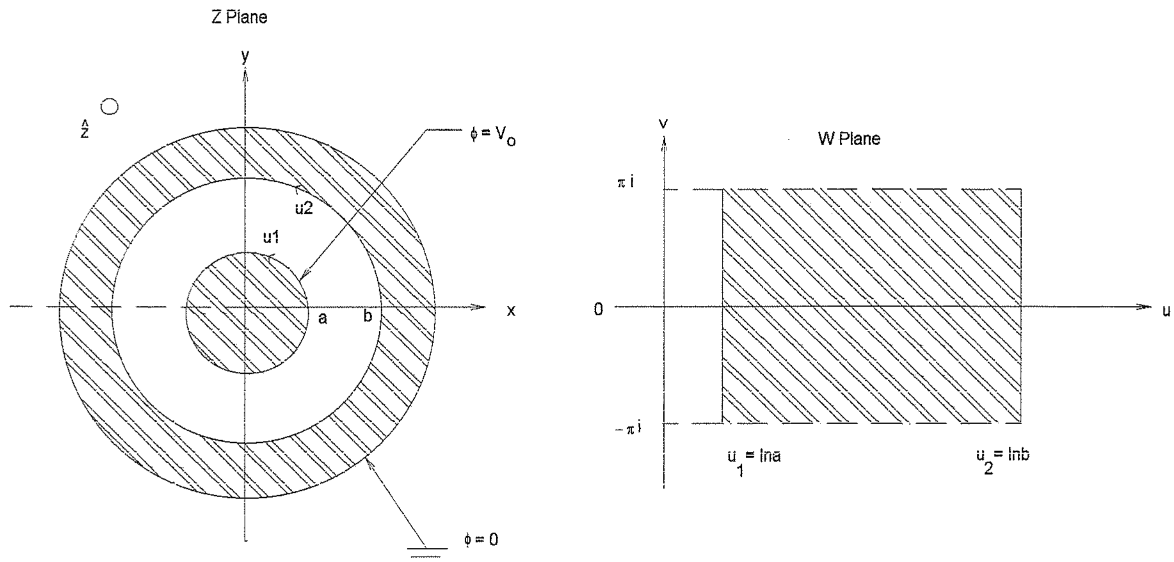


Figure 8-2-1: Cross-section of the Transmission Line and its Mapping

We have $\vec{E}_0 = -\nabla_2\phi$ and $\Delta_2\phi = 0$. Thus ϕ is harmonic in the x-y plane, and therefore forms the real part of the analytic function:

$$\Omega = \phi + i\Psi \quad \text{where on } u_1,\ \phi = V_0 \qquad 8\text{-}2\text{-}1$$

$$u_2,\ \phi = 0 \qquad 8\text{-}2\text{-}2$$

Ω obeys the Cauchy-Riemann equations so that

$$
\begin{aligned}
0 &= \left(\partial_x + i\partial_y\right)\left(\phi + i\Psi\right) \\
&= \left(\partial_x\phi - \partial_y\Psi\right) + i\left(\partial_y\phi + \partial_x\Psi\right) \\
&= \left(\partial_x + i\partial_y\right)\phi + i\left(\partial_x + i\partial_y\right)\Psi \\
-\left(\partial_x + i\partial_y\right)\phi &= -i\left[-\left(\partial_x + i\partial_y\right)\Psi\right]
\end{aligned}
\qquad \text{8-2-3}
$$

Now this conforms to $\vec{E}_0 = -\nabla\phi = -\left(\partial_x\phi\,\hat{i} + \partial_y\phi\,\hat{j}\right) \leftrightarrow -\left(\partial_x\phi + i\partial_y\phi\right)$ if we associate complex numbers with vectors in the x-y plane as follows $x\hat{i} + y\hat{j} \leftrightarrow x + iy$. Then the significance of Ψ is:

$$
\begin{aligned}
-\left(\partial_x + i\partial_y\right)\Psi &= i\left[-\left(\partial_x + i\partial_y\right)\phi\right] \\
&= iE_0 \\
&= \exp\left(i\tfrac{\Pi}{2}\right)E_0 \\
&= B_0
\end{aligned}
\qquad \text{8-2-4}
$$

due to the rotation of 90 degrees so that we can interpret Ψ as $\vec{B} = -\nabla_2\Psi$. 8-2-5

Now we see that the conformal mapping will solve for the magnetic and electric components of the wave's field.

Thus the electromagnetic field in the annulus between the conductors of a transmission line is in the form of a wave of the form:

$$F = i\left(\cos k_0(t - z)\right)\left(1 + i\hat{k}\right)\vec{E}_0 \qquad \text{8-2-6}$$

where $\vec{E}_0 = -\nabla_2\phi$ and ϕ satisfies Laplace's equation $\Delta_2\phi = 0$ in the annulus.

Now a solution to the transmission line problem will be explored using the conformal mapping. For this the most simplest case will be explored.

8-3 Transmission Line with Concentric Circular Cross-Section (using Conformal Mapping)

Here we take a coaxial line with a concentric circular cross-section with a vacuum in the annulus. This provides the simplest case for this application. Refer to figure 8-3-1

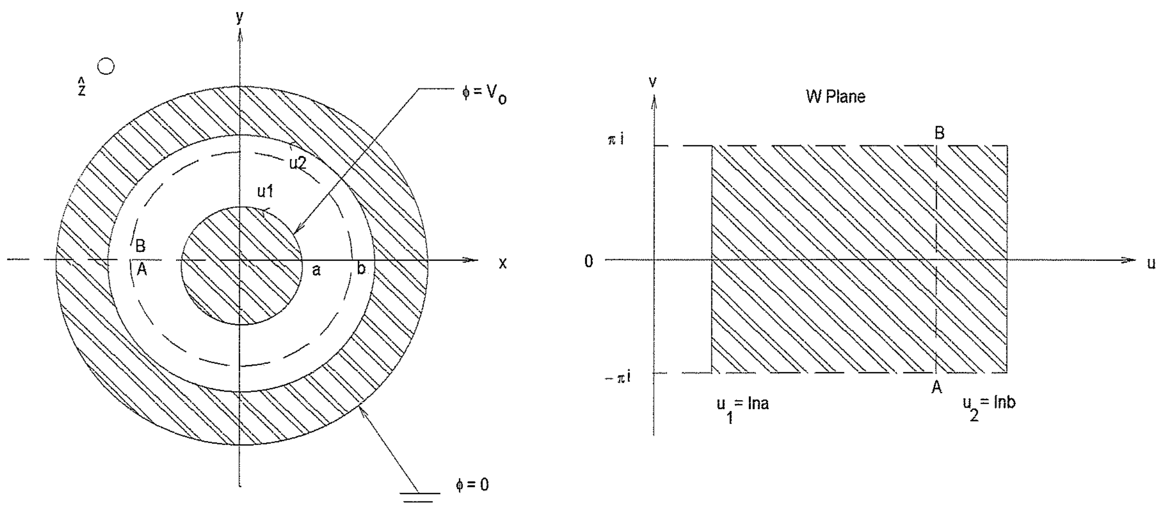


Figure 8-3-1

The mapping that will accomplish this is

$$
\begin{aligned}
z &= \exp w \\
&= \exp(u+iv) \\
&= (\exp u)(\cos v + i\sin v)
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
w &= u+iv = \log z \\
&= \ln r + i\theta \\
&-\pi < \theta \le \pi
\end{aligned}
\qquad 8\text{-}3\text{-}1
$$

so $x = \exp(u)\cos v$ and $y = \exp(u)\sin v$ also $u = \ln r$. Note the branch cut on the negative x-axis. The complex potential is then $\Omega = \phi + i\Psi = Aw + B = A(u+iv) + B$ 8-3-2

On the boundaries: u1 $\phi = V_0 \longrightarrow Au_1 + B = V_0$ 8-3-3

u2 $\phi = 0 \longrightarrow Au_2 + B = 0$ 8-3-4

Then this implies:

$$A = \frac{V_0}{(u_1 - u_2)}$$
$$B = -Au_2 = -\frac{V_0 u_2}{(u_1 - u_2)} \qquad \text{8-3-5}$$

So then

$$\Omega = \frac{V_0}{(u_1 - u_2)}(u + iv) - \frac{V_0 u_2}{(u_1 - u_2)}$$
$$= \frac{V_0}{(u_1 - u_2)}(u - u_2 + iv) \qquad \text{8-3-6}$$
$$= \frac{V_0}{(u_1 - u_2)}(w - u_2)$$

So the field potential is $\phi = \mathrm{Re}(\Omega) = \frac{V_0(u - u_2)}{u_1 - u_2} = \frac{V_0(\ln r - \ln b)}{\ln a - \ln b} = V_0 \frac{\ln \frac{r}{b}}{\ln \frac{a}{b}}$. 8-3-7

So the potential difference between the conductors is $\delta\phi = V_0 \left(\frac{\ln \frac{a}{b} - \ln \frac{b}{b}}{\ln \frac{a}{b}} \right) = V_0$ 8-3-8

Which is expected for a coaxial cable.

Now the current in the central conductor can be found by taking a simple path, S, from one edge of the branch cut anti-clockwise to the other side of the branch cut (i.e. point A to B). This is necessary as a closed path would cut the branch cut and give us zero current. We can then use the definition of current namely

$$
\begin{aligned}
I &= \int_S \vec{J} \circ \hat{k} dS \\
&= \int_S \vec{J} \circ d\vec{S} \qquad \text{since } \vec{E} \perp \hat{k} \qquad \text{8-3-9} \\
&= \int_S \left(\vec{J} + \frac{\partial \vec{E}}{\partial t} \right) \circ d\vec{S}
\end{aligned}
$$

where S is a plane surface perpendicular to $\hat{k}$.

Now if we use Maxwell's third equation and Stokes theorem we can evaluate this result

$$
\begin{aligned}
I &= \int_S \left(\vec{J} + \frac{\partial \vec{E}}{\partial t} \right) \circ d\vec{S} \\
&= \int_S \left(\nabla \times \vec{B} \right) \circ d\vec{S} \\
&= \oint_C \vec{B} \circ d\vec{r} \\
&= -\oint_C \nabla \Psi \circ d\vec{r} \qquad \text{8-3-10} \\
&= -\int_A^B d\Psi \\
&= -\left(\Psi_B - \Psi_A \right) \\
&= -\delta \Psi
\end{aligned}
$$

Now it is clear why the branch cut is really necessary. If we use a closed path in a region in which Ω is analytic we would have $I = -\left(\Psi_A - \Psi_A \right) = 0$ which does not make sense physically.

We need to evaluate:

$$
\Psi = \mathrm{Im}(\Omega) = \frac{V_0}{u_1 - u_2} v \qquad \text{8-3-11}
$$

$$\delta\Psi = \Psi_B - \Psi_A = \frac{V_0}{u_1 - u_2}(v_B - v_A)$$

$$= \frac{V_0}{u_1 - u_2}(\pi - (-\pi)) \qquad \text{8-3-12}$$

$$= \frac{2\pi V_0}{u_1 - u_2}$$

So the current is $I_0 = -\delta\Psi = \frac{2\pi V_0}{u_2 - u_1} = \frac{2\pi V_0}{\ln b - \ln a} = \frac{2\pi V_0}{\ln \frac{b}{a}}$. 8-3-13

In full space and time the wave is as follows:

$$F = i(\cos k_0(t - z))(1 + i\hat{k})\vec{E}_0$$

$$= i(\cos k_0(t - z))(1 + i\hat{k})(-\nabla_2 \phi) \qquad \text{8-3-14}$$

$$= i(\cos k_0(t - z))(1 + i\hat{k})\left(\frac{V_0}{r \ln \frac{b}{a}}\hat{r}\right)$$

where $\hat{r}$ is the unit radial vector perpendicular to $\hat{k}$.

Also $I = \frac{2\pi V_0}{\ln \frac{b}{a}}(\cos k_0(t - z))$, varying in time. 8-3-15

The impedance is $Z_o = \frac{V_0}{I_0} = \frac{\delta\phi}{-\delta\Psi} = \frac{V_0}{\frac{2\pi V_0}{\ln \frac{b}{a}}} = \frac{\ln \frac{b}{a}}{2\pi}$ 8-3-16

It is now clear that the method of conformal mapping is the superior way of solving this type of boundary value problem using quaternionic forms.

8-4 Transmission Line with a Confocal Ellipsoidal Cross-Section

In this case the cross-section of the coaxial line is assumed to be concentric ellipsoids. Again we need a mapping to the w-plane from the z-plane. Refer to Figure 8-4-1. C_0 represents the foci of the ellipses.

One such mapping that will work is

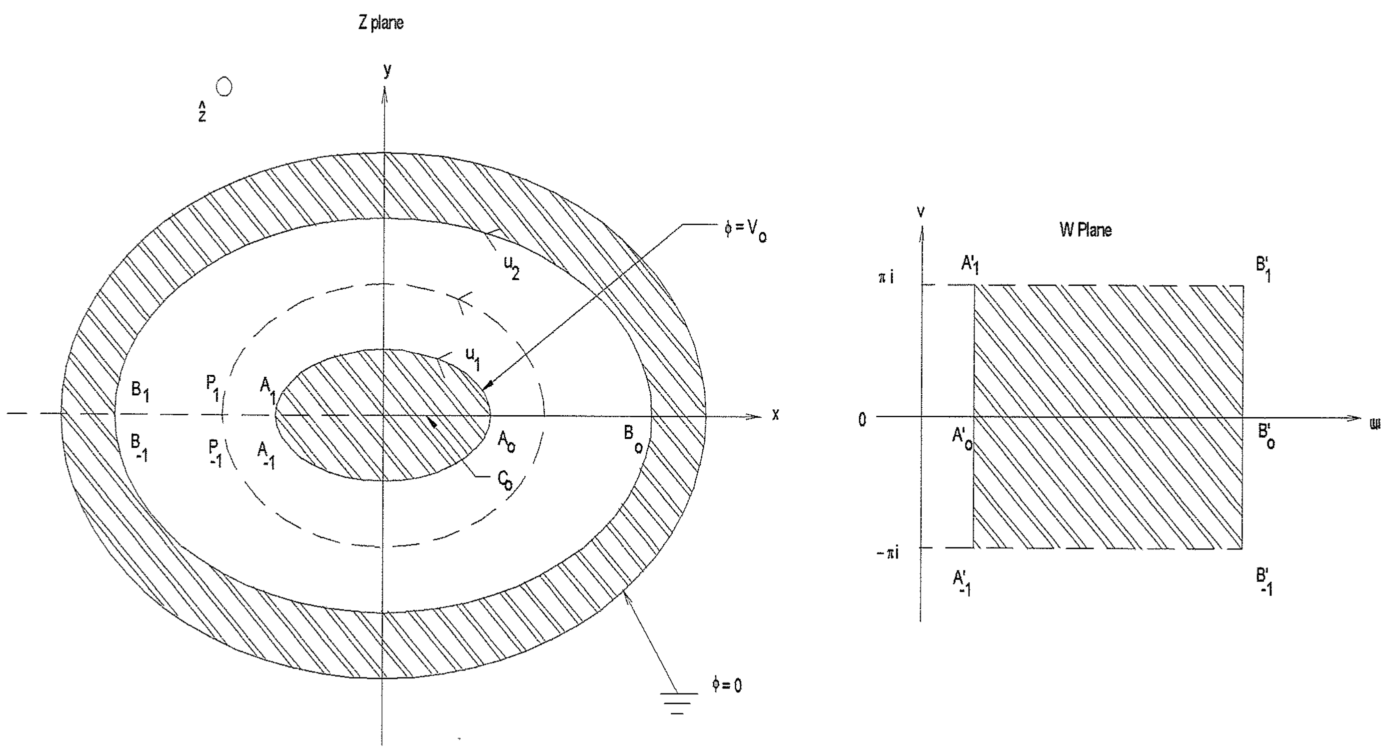


Figure 8-4-1: Cross-section of a Transmission Line assumed to be Ellipsoidal

$$\begin{aligned} z &= C_0 \cosh w = C_0 \cosh(u + iv) \\ &= C_0[\cosh u \cosh(iv) + \sinh u \sinh(iv)] \\ &= C_0[\cosh u \cos v + i \sinh u \sin v] \\ &= [x + iy] \end{aligned} \qquad \text{8-4-1}$$

If we take a constant u = u_1 then

$$x = a_1 \cos v \qquad \text{where} \qquad a_1 = C_0 \cosh u$$
$$y = b_1 \sin v \qquad\qquad\qquad b_1 = C_0 \sinh u \qquad\qquad 8\text{-}4\text{-}2$$

so $\dfrac{x^2}{a_1^2} + \dfrac{y^2}{b_1^2} = 1$ indeed an ellipse 8-4-3

and $a_1^2 - b_1^2 = C_0^2\left(\cosh^2 u - \sinh^2 u\right) = C_0^2$ 8-4-4

so C_0 is indeed the focus of the ellipses.

If we look at a constant v = v_1 then

$$x = a_2 \cos u$$
$$y = b_2 \sin u \qquad \text{which form hyperbolae} \qquad 8\text{-}4\text{-}5$$

Thus constant u forms ellipses and constant v form hyperbolae that intersect at right angles to the ellipses.

We can now find our complex potential $\Omega = \phi + i\Psi = Aw + B = A(u + iv) + B$ 8-4-6

Here on u = u2, $\phi = 0 \quad Au_2 + B = 0$ 8-4-7

u = u1, $\phi = V_0 \quad Au_1 + B = V_0$ 8-4-8

So that we find:

$$A = \frac{V_0}{(u_1 - u_2)}$$
$$B = -Au_2 = -\frac{V_0 u_2}{(u_1 - u_2)} \qquad 8\text{-}4\text{-}9$$

So then $\Omega = \frac{V_0}{(u_1 - u_2)}(w - u_2)$ 8-4-10

which is the same solution obtained for the case of the concentric circles.

$$(u + iv) = \frac{(u_1 - u_2)}{V_0}(\phi + i\Psi) + u_2 = \left(u_2 - \frac{(u_2 - u_1)}{V_0}\phi\right) - i\frac{(u_2 - u_1)}{V_0}\Psi \quad 8\text{-}4\text{-}11$$

Now referring to equation 8-4-1:

$$x = C_0 \cosh u \cos v \qquad\qquad y = C_0 \sinh u \sin v$$

$$= C_0 \cosh\left(u_2 - \frac{\delta u}{V_0}\phi\right)\cos\left(\frac{\delta u}{V_0}\Psi\right) \text{ and } = -C_0 \sinh\left(u_2 - \frac{\delta u}{V_0}\phi\right)\sin\left(\frac{\delta u}{V_0}\Psi\right) \quad 8\text{-}4\text{-}12$$

This gives x and y in terms of the potentials ϕ and Ψ.

Now the current in the central conductor can be found in the same way as in section 8-3. As before:

$$I = -\delta\Psi \quad 8\text{-}4\text{-}13$$

Once again:

$$\Psi = \frac{V_0}{u_1 - u_2}v \quad 8\text{-}4\text{-}14$$

$$\delta\Psi = \frac{V_0}{u_1 - u_2}(v_B - v_A) = \frac{2\pi V_0}{u_1 - u_2} \quad 8\text{-}4\text{-}15$$

So the current is $I_0 = -\delta\Psi = \frac{2\pi V_0}{u_2 - u_1}$. 8-4-16

In space and time the field is:

$$F = i\left(\cos k_0(t-z)\right)\left(1+i\hat{k}\right)\left(-\nabla_2\phi\right) \qquad \text{8-4-17}$$

$$\text{where } \nabla_2\phi = \frac{\partial\bar{r}/\partial\phi}{\left|\partial\bar{r}/\partial\phi\right|^2} \qquad \text{8-4-18}$$

[see Diamant, p 201]

where $\bar{r} = x\hat{i} + y\hat{j}$.

$$\text{The impedance is } Z_o = \frac{V_0}{I_0} = \frac{\delta\phi}{-\delta\Psi} = \frac{u_2 - u_1}{2\pi} \qquad \text{8-4-19}$$

8-5 Transmission Line with a Concentric Circular Cross-Section and a Dielectric-Filled Annulus

We need to first find a solution to the wave transmission problem using the modified Maxwell equation. Recall equation 6-3-2:

$$\mathbf{DF}^* = \mathbf{J} \qquad \text{or } \mathbf{D}^*\mathbf{F} = \mathbf{J}^* \quad \text{(see sections 6-2 and 6-3)} \qquad \text{8-5-1}$$

Assume, in keeping with the previous results, a form for $\mathbf{F}$ in the annulus as follows

$$\mathbf{F} = i\left(\cos k_0(\tau - z)\right)\left(1 + i\hat{k}\right)\vec{E}_0 \qquad \text{8-5-2}$$

where $\vec{E}_0 = -\nabla_2 \phi$ as before. 8-5-3

Then exactly as before $\mathbf{DF}^* = 0$. 8-5-4

Now we use this modified result to solve for the circular cross-section with a dielectric in the annulus exactly as before. The problem is represented in Figure 8-5-1.

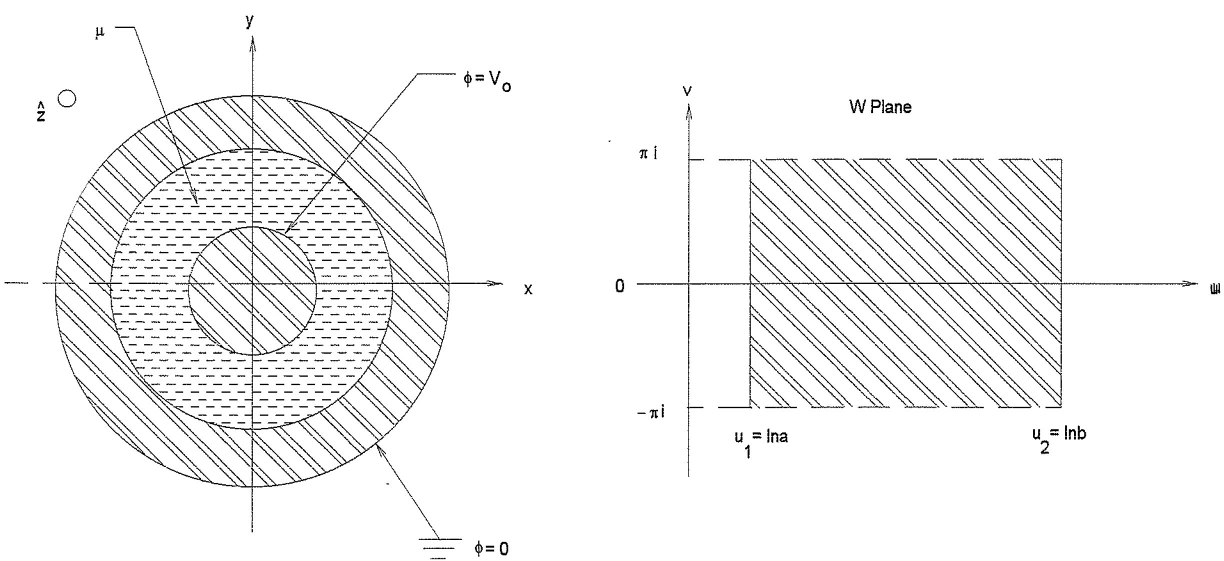


Figure 8-5-1: The Dielectric Filled Transmission Line

Thus we have

$$z = (\exp u)(\cos v + i \sin v) \text{ or } \qquad w = \ln r + i\theta \quad -\pi < \theta \leq \pi \qquad 8\text{-}5\text{-}5$$

$$\Omega = \phi + i\Psi = Aw + B = A(u + iv) + B \qquad 8\text{-}5\text{-}6$$

On the boundaries: u = u1, $\phi = V_0$ 8-5-7

u = u2, $\phi = 0$ 8-5-8

So

$$A = \frac{V_0}{(u_1 - u_2)}$$
$$B = -\frac{V_0 u_2}{(u_1 - u_2)} \qquad 8\text{-}5\text{-}9$$

Again we have $\Omega = \frac{V_0}{(u_1 - u_2)}(u - u_2 + iv)$ 8-5-10

In order to find the current in the central conductor, first note that Maxwell's third equation (M3) now reads $\nabla \times \vec{G} - \partial_\tau \vec{E} = \frac{1}{\in c}\vec{J}$. Also let us define the local impedance as $\eta = \sqrt{\frac{\mu}{\in}} = \lambda\mu = \frac{1}{\lambda \in}$. Now take a simple path, S, from one edge of the branch cut anti-clockwise to the other side of the branch cut.

Then

$$
\begin{aligned}
I &= \int_S \vec{J} \circ d\vec{S} \\
&= \int_S \left(\vec{J} + \in c \frac{\partial \vec{E}}{\partial \tau} \right) \circ d\vec{S} \\
&= \in c \int_S \left(\nabla \times \vec{G} \right) \circ d\vec{S} \\
&= \in c \oint_C \vec{G} \circ d\vec{r} \\
&= - \in c \oint_C \nabla \Psi \circ d\vec{r} \\
&= - \in c \delta \Psi
\end{aligned}
\qquad 8\text{-}5\text{-}12
$$

where $\in c = \frac{c_0}{\eta}$ in Heaviside-Lorentz units and $\in c = \frac{1}{\eta}$ in SI units. 8-5-13

The current is $I_0 = - \in c \delta \Psi = \in c \frac{2\pi V_0}{u_2 - u_1} = \in c \frac{2\pi V_0}{\ln \frac{b}{a}}$ 8-5-14

The impedance is $Z_o = \frac{1}{\in c} \frac{\ln \frac{b}{a}}{2\pi}$ 8-5-15

So that in Heaviside-Lorentz units $Z_o = \frac{\eta}{2\pi c_0} \ln \frac{b}{a}$ and in SI $Z_o = \frac{\eta}{2\pi} \ln \frac{b}{a}$ 8-5-16

The field in space time is :

$$
\begin{aligned}
\mathbb{F} &= i\left(\cos k_0 (\tau - z)\right)\left(1 + i\hat{k}\right)\left(- \nabla_2 \phi\right) \\
&= i\left(\cos k_0 (\tau - z)\right)\left(1 + i\hat{k}\right)\left(\frac{V_0}{r \ln \frac{b}{a}} \hat{r}\right)
\end{aligned}
\qquad 8\text{-}5\text{-}17
$$

where $\hat{r}$ is the unit radial vector perpendicular to $\hat{k}$.

Also $I = \frac{2\pi V_0}{\ln \frac{b}{a}} \left(\cos k_0 (\tau - z) \right)$, varying in time. 8-5-17

8-6 Transmission Line with Concentric Ellipsoidal Cross-Section and Dielectric-Filled Annulus

The solution in section 8-5, namely $\mathbb{F} = i(\cos k_0(\tau - z))(1 + i\hat{k})\vec{E}_0$, with $\vec{E}_0 = -\nabla_2\phi$, remains valid in this case, where now ϕ is the solution for the ellipsoidal case as in section 8-4.

Thus $z = C_0[\cosh u \cos v + i \sinh u \sin v] = [x + iy]$ 8-6-1

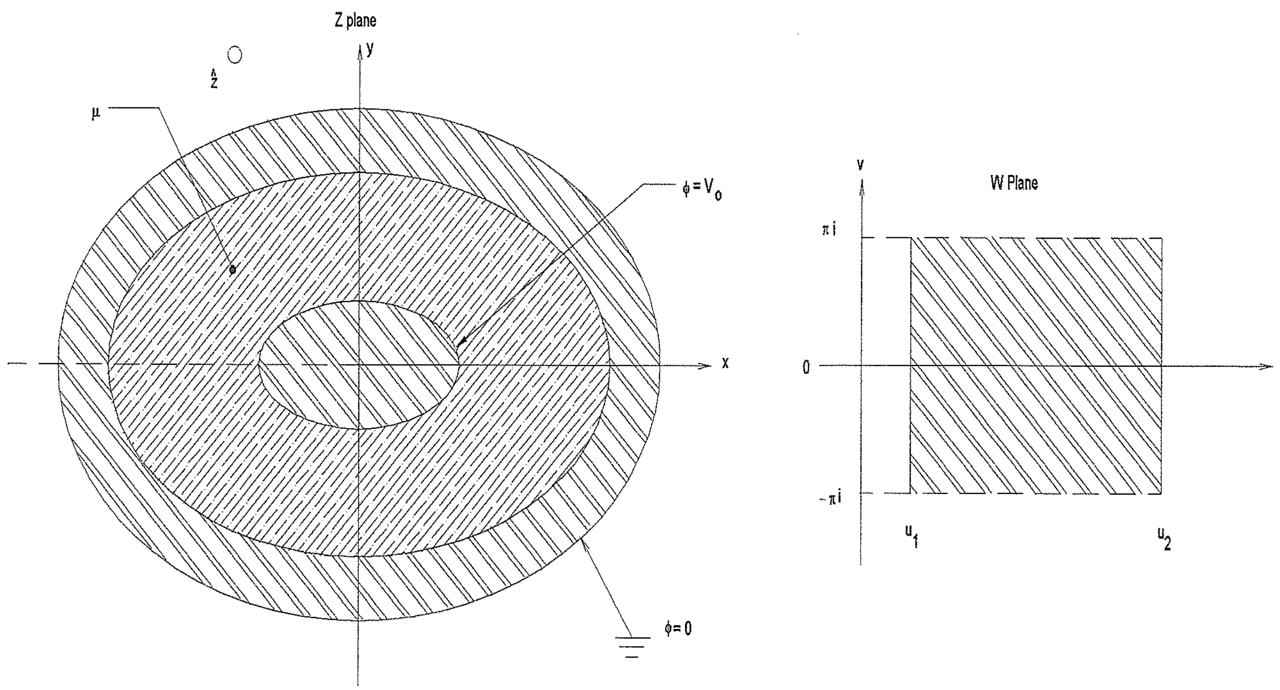


Figure 8-6-1: Dielectric Filled Ellipsoidal Transmission Line

$$\Omega = \phi + i\Psi = Aw + B = A(u + iv) + B \qquad 8\text{-}6\text{-}2$$

As before we have this solution:

$$A = \frac{V_0}{(u_1 - u_2)}$$
$$B = -\frac{V_0 u_2}{(u_1 - u_2)} \qquad \text{8-6-3}$$

So then $\Omega = \frac{V_0}{(u_1 - u_2)}(w - u_2)$ 8-6-4

$$(u + iv) = \frac{(u_1 - u_2)}{V_0}(\phi + i\Psi) + u_2 = \left(u_2 - \frac{(u_2 - u_1)}{V_0}\phi\right) - i\frac{(u_2 - u_1)}{V_0}\Psi \qquad \text{8-6-5}$$

Now referring to equation 8-4-1:

$$x = C_0 \cosh u \cos v \qquad y = C_0 \sinh u \sin v$$
$$= C_0 \cosh\left(u_2 - \frac{\delta u}{V_0}\phi\right)\cos\left(\frac{\delta u}{V_0}\Psi\right) \text{ and } = -C_0 \sinh\left(u_2 - \frac{\delta u}{V_0}\phi\right)\sin\left(\frac{\delta u}{V_0}\Psi\right) \qquad \text{8-6-6}$$

This gives x and y in terms of the potentials ϕ and Ψ.

Once again:

$$\Psi = \frac{V_0}{u_1 - u_2}v \qquad \text{8-6-7}$$

$$\delta\Psi = \frac{2\pi V_0}{u_1 - u_2} \qquad \text{8-6-8}$$

Now the current in the central conductor can be found in the same way as in section 8-3. As before:

$$I_0 = -\in c\delta\Psi \qquad \text{8-6-9}$$

where $\in c = \frac{c_0}{\eta}$ in Heaviside-Lorentz units 8-6-10 and $\in c = \frac{1}{\eta}$ in SI units. 8-6-11

The current is $I_0 = -\in c\delta\Psi = \in c\frac{2\pi V_0}{u_2 - u_1} = \in c\frac{2\pi V_0}{\delta u}$ 8-6-12

In space and time the field is:

$$\mathbb{F} = i(\cos k_0(\tau - z))(1 + i\hat{k})(-\nabla_2\phi)$$ 8-6-13

where $\nabla_2\phi = \frac{\partial\bar{r}/\partial\phi}{\left|\partial\bar{r}/\partial\phi\right|^2}$ 8-6-14

[see Diamant, p 201]

where $\bar{r} = x\hat{i} + y\hat{j}$.

The time varying current is $I = \in c\frac{2\pi V_0}{\delta u}(\cos k_0(\tau - z))$ 8-6-15

The impedance is given as follows:

in Heaviside-Lorentz units $Z_o = \frac{\eta\delta u}{2\pi c_0}$ 8-6-16

and in SI $Z_o = \frac{\eta\delta u}{2\pi}$ 8-6-17

This chapter illustrates the practical use of quaternions in the field of electromagnetism.

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